

Towards the saturation of the Froissart bound

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It is the aim of this paper to review the constructions of pion-pion scattering amplitudes that rigorously satisfy Mandelstam analyticity, crossing symmetry, and (at least partly) the constraints imposed by elastic and inelastic unitarity. Three types of amplitudes are considered in detail: amplitudes that are given by a Mandelstam representation, analytic function defined by an explicit Regge type ansatz, and amplitudes with Regge poles in the Khuri or the Watson-Sommerfeld representation. The results are discussed under particular emphasis of a strong increase of the absorptive part of the forward amplitude and the saturation of the Froissart bound. Demanding all constraints the optimal construction obtained so far yields (via the optical theorem) a total cross section, which decreases like $(\log E)^{-3}$, where E is the energy of the scattering process. The increasing cross section of the Froissart bound has been saturated by amplitudes, which satisfy analyticity, crossing symmetry and the constraints imposed by inelastic unitarity; but elastic unitarity is missing. The problems caused by elastic unitarity are discussed in detail.

1 Introduction

One of the outstanding results of the analytic S-matrix theory is the Froissart bound

$$\sigma_{tot}(s) \leq const (\log s)^2 \quad (1)$$

for the total cross section of a two particle scattering process, where s is the square of the centre of mass energy. This bound has been derived 1961 by Froissart [Fro61] assuming that the two particle scattering amplitude has uniformly bounded partial wave amplitudes and satisfies a Mandelstam representation with a finite number of subtractions. Then Martin [Mar63, Mar66] has established this bound using only the analyticity domain of axiomatic quantum field theory and positivity properties of the absorptive part. In the meantime experimental results [UA493] indicate an increase of the total pp cross section, which is compatible with a $(\log s)^2$ behaviour, and future experiments at BNL-RHIC and CERN-LHC may confirm this increase [BSW03]. The derivation of the Froissart bound (1) follows from only a part of the analyticity and unitarity properties, which can be formulated with the elastic two particle scattering amplitude; and it is still an open problem, whether the Froissart bound can be improved, if all these constraints are taken into account. For the scattering process with the strongest crossing restrictions – the scattering of neutral (or isospin-1) pions – the existence of amplitudes, which satisfy elastic unitarity and the unitarity inequalities in the inelastic regime, has been derived in 1968 by Atkinson [Atk68a, Atk68b]. Thereby elastic unitarity is incorporated by a non-linear fixed point mapping. The final construction using a Mandelstam representation with one subtraction has lead to an amplitude which allows an asymptotic behaviour

$$\sigma_{tot}(s) \sim (\log s)^{-3} \text{ for } s \rightarrow \infty \quad (2)$$

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of the total cross section [Atk70]. So far there is no proof of the existence of an amplitude, which comes closer to the Froissart bound, if all constraints are rigorously fulfilled. But if one does not demand elastic unitarity, amplitudes have been constructed, which saturate the Froissart bound [Kup82].

In the main part of this paper we give a review about the construction of amplitudes that rigorously satisfy Mandelstam analyticity, the crossing symmetry of neutral pions and – at least to some extent – the unitarity constraints, which can be formulated with the elastic scattering amplitude alone. The discussion concentrates on scattering amplitudes with an optimal increase of the absorptive part of the forward amplitude at high energies. In addition to that we investigate the problems, which originate from elastic unitarity, if one wants to obtain a better result than (2). All results can be easily extended to the scattering of isospin-1 pions.

The paper is organized as follows. The notations and the assumptions about analyticity, crossing and unitarity are given in section 2. In section 3 we recapitulate a few results on upper bounds, which are important for the subsequent sections. Then we discuss in some detail three types of amplitudes, which have been investigated in the literature:

(i) Amplitudes represented by the Mandelstam spectral integrals

A non-linear fixed-point mapping has been developed to incorporate elastic unitarity for amplitudes of this type [Atk68a, Atk68b, Atk69, Atk70, Kup69]. If the Mandelstam representation has at most one subtraction and the spectral functions are positive, it is possible to incorporate also the inelastic unitarity constraints. The result (2) has been obtained for an amplitude with one subtraction. The construction of these amplitudes and the limitation to (2) are presented in section 4.

(ii) Analytic functions with a Regge type asymptotics

Starting with an explicit ansatz one can obtain analytic functions with a Regge type asymptotics that satisfy Mandelstam analyticity, crossing symmetry and the constraints of inelastic unitarity. The first solutions have been obtained for amplitudes with simple Regge poles or double poles [Kup71]. Then the total cross section behaves like

$$\sigma_{tot}(s) \sim s^{\alpha(0)-1} (\log s)^n \text{ for } s \rightarrow \infty \quad (3)$$

with an intercept $\alpha(0) \leq 1$ of the leading Regge trajectory and an exponent $n = 0$ or 1 of the logarithm. The validity of the constraints of inelastic unitarity is derived using a linearization of the quadratic unitarity inequalities. This technique of linearization is recapitulated in section 5.1 and the construction of the amplitudes with Regge poles is presented in section 5.2.

These methods have been extended in [Kup82] to amplitudes, which have crossing Regge cuts and which saturate the Froissart bound. The main step for this construction is indicated in section 5.3.

The amplitudes of section 5 do not satisfy elastic unitarity. Already in 1960 Gribov [Gri60, Gri61] realized that Mandelstam analyticity, crossing symmetry and elastic unitarity impose irritating constraints on the high energy behaviour of scattering amplitudes. The construction – or the mere proof of existence – of Regge amplitudes, which satisfy crossing symmetry and elastic unitarity is therefore a non-trivial task. For that purpose we consider in section 6

(iii) Amplitudes in the Khuri representation

To impose elastic unitarity one can generalize the non-linear fixed point mapping to Regge amplitudes using the Watson-Sommerfeld transform [AFJK76] or the Khuri representation

[Kup77]. It is possible to obtain solutions with one rising Regge trajectory that satisfy crossing symmetry and elastic unitarity and that have an asymptotically constant total cross-section, see section 6.1. The partial wave amplitudes of these solutions are uniformly bounded for all energies, but the validity of the additional constraints of inelastic unitarity remains questionable. In section 6.2 we discuss the present status of this problem.

As already mentioned the optimal result, which has been obtained assuming all crossing and unitarity constraints, is (2). Extending the methods presented in sections 5.2 and 6.1 one might succeed satisfying all constraints to construct an amplitude with an asymptotically constant total cross section. But to obtain solutions with increasing total cross sections one has to develop new techniques. The puzzle of the Froissart bound is not yet settled.

The problem of the saturation of the Froissart bound has also been investigated assuming crossing symmetry in the smaller analyticity domain of the axiomatic quantum field theory [Khu76]. Unfortunately these methods have not lead to a conclusive answer and will not be discussed in this paper.

The details of the constructions of the amplitudes can be found in the cited literature. But for completeness some calculations are given in the Appendices A – C.

2 Notations and basic assumptions

The kinematics is always the kinematics of pion scattering with the Mandelstam variables s , t and u . The mass of the particles is normalized to unity such that $s + t + u = 4$. We consider the construction of amplitudes with the following properties:

1. The amplitude $A(s, t)$ is holomorphic in the Mandelstam domain

$$\mathbb{C}_{cut}^2 = \{(s, t) \in \mathbb{C}^2 \mid s \notin [4, \infty), t \notin [4, \infty), s + t \notin (-\infty, 0]\}. \quad (4)$$

2. The amplitude $A(s, t)$ is polynomially bounded

$$|A(s, t)| \leq \text{const} (1 + |s| + |t|)^n \quad (5)$$

for $(s, t) \in \mathbb{C}_{cut}^2$ with some positive number n , and the boundary values of $A(s, t)$ are Hölder continuous with an index $\mu \in (0, \frac{1}{2}]$.

3. If not stated otherwise, we consider neutral pions, and crossing symmetry means:
The amplitude is symmetric in the variables s , t and u

$$A(s, t) = A(t, s) = A(s, u) = A(u, s) = A(u, t) = A(t, u). \quad (6)$$

A more precise form of Mandelstam analyticity and crossing symmetry is the following. Let $F(s, t)$ be a polynomially bounded function which is analytic in the domain

$$\mathbb{D} = \{(s, t) \in \mathbb{C}^2 \mid s \notin [4, \infty), t \notin [16, \infty)\}, \quad (7)$$

then $A(s, t) = \text{Sym } F(s, t)$ with

$$\text{Sym } F(s, t) := F(s, t) + F(t, s) + F(s, u) + F(u, s) + F(u, t) + F(t, u) \quad (8)$$

is an amplitude, which satisfies Mandelstam analyticity – as introduced in [Man58] – and crossing symmetry. The amplitudes constructed in the following sections have this form. But an extension

to the isospin-1 crossing symmetry of charged pions is easily possible, as can be seen from the cited literature.

The absorptive part in the s -channel is

$$A_s(s, t) \equiv \text{Abs}_s A(s, t) := (2i)^{-1} (A(s + i0, t) - A(s - i0, t)), \quad s \geq 4;$$

it agrees with the imaginary part $\text{Im } A(s + i0, t)$ if $-s < t < 4$. The amplitude is normalized such that the total cross section is given by

$$\sigma_{total}(s) = \frac{8\pi}{\sqrt{s(s-4)}} A_s(s, 0) = \frac{8\pi}{\sqrt{s(s-4)}} \text{Im } A(s + i0, 0) \quad (9)$$

The partial wave expansion in the physical domain is

$$A(s + i0, t) = 2\sqrt{\frac{s}{s-4}} \sum_l (2l+1) a_l(s) P_l(z), \quad s \geq 4 \quad (10)$$

with $z = 1 + \frac{2t}{s-4}$. The partial waves are given by

$$a_l(s) = \frac{1}{2} [s(s-4)]^{-\frac{1}{2}} \int_{4-s}^0 A(s + i0, t) P_l(z) dt, \quad l = 0, 1, 2, \dots \quad (11)$$

For $l > n$ this integral is equivalent to the Froissart-Gribov integral

$$a_l(s) = \frac{2}{\pi} [s(s-4)]^{-\frac{1}{2}} \int_4^\infty A_t(s + i0, t) Q_l\left(1 + \frac{2t}{s-4}\right) dt. \quad (12)$$

Due to the crossing symmetry (6) all odd partial wave amplitudes vanish.

The amplitudes should satisfy at least partly the following unitarity constraints, which can be formulated with the two-particle scattering amplitude alone:

4. The elastic unitarity identities

$$\text{Im } a_l(s) = |a_l(s)|^2 \quad \text{for } l = 0, 1, 2, \dots, \text{ and } 4 \leq s \leq 16. \quad (13)$$

5. The inelastic unitarity inequalities

$$1 \geq \text{Im } a_l(s) \geq |a_l(s)|^2 \quad \text{for } l = 0, 1, 2, \dots \quad (14)$$

and energies $s \geq 16$. In the sequel we also consider amplitudes, which fulfill the inequalities (14) for all $s \geq 4$, but elastic unitarity does not hold.

The threshold behaviour of amplitudes, which satisfy the inequalities (14) but not elastic unitarity, can be rather arbitrary. The constructions presented in this paper satisfy the following uniform estimate, which is compatible with elastic unitarity:

6. The absorptive part in the s -channel has the bound

$$|A_s(s, t)| \leq \text{const} \sqrt{\frac{s-4}{s}} (1 + |s| + |t|)^n \quad (15)$$

for $s \geq 4$ and $t \in \mathbb{C}_{cut} = \mathbb{C} \setminus (-\infty, -s] \cup [4, \infty)$ including the boundary values at $t \pm i0$ if $t \geq 4$ or $t \leq -s$.

Finally we give some mathematical notations used in this paper. A number $s \in \mathbb{R}$ is called *positive* if $s \geq 0$ and *strictly positive* if $s > 0$. For a real variable s and a complex parameter λ the function $\mathbb{R} \ni s \rightarrow s_+^\lambda \in \mathbb{C}$ is defined as $s_+^\lambda = 0$ if $s \leq 0$ and $s_+^\lambda = s^\lambda$ if $s > 0$. The hat in $\hat{F}(s, t)$ indicates that the function is symmetric with respect to its variables, $\hat{F}(s, t) = \hat{F}(t, s)$.

3 Restrictions for the total cross section

In this section we shortly recapitulate bounds and constraints of the absorptive part of the forward amplitude. From general principles follows that the boundary values $A(s+i0, t)$, $s \geq 4$, are integrable functions, which have no meaning pointwise. Bounds like the Froissart bound are therefore bounds for local averages, see e. g. section 17.1 of [BLOT90]. But for amplitudes with uniformly continuous boundary values – as considered in this paper – the local statements can be used.

3.1 The Froissart bound

The bound

$$\text{Im } A(s+i0, 0) \leq \text{const } s (\log s)^2, \quad s \geq 4, \quad (16)$$

has first been derived by Froissart [Fro61] assuming the Mandelstam representation with a finite number of subtraction and bounded partial waves. Then Martin [Mar63, Mar66, Mar69b] has succeeded to derive this bound from the analyticity domain of axiomatic quantum field theory and the unitarity constraints

$$0 \leq \text{Im } a_l(s) \leq 1, \quad l = 0, 1, 2, \dots, \quad s \geq 4. \quad (17)$$

The Froissart bound is therefore a consequence of only a small part of the analyticity and unitarity constraints of a two-particle scattering amplitude. Especially, it is valid without assumptions about the crossed channels. As already mentioned, it is possible to saturate this bound with an amplitude, which satisfies the constraints 1. – 3. and 5. of section 2. But elastic unitarity is missing in this construction; see section 5.3.

3.2 A bound for amplitudes with positive spectral functions

For the construction – or for the mere proof of existence – of amplitudes, which satisfy elastic unitarity, analytic functions with positive spectral functions turn out to be of exceptional importance; see the detailed discussion in section 4.1. It is therefore of some interest to know that the positivity of spectral functions leads to a more restrictive bound of the amplitudes. The following bound has been derived by Goebel [Goe61] and by Martin [Mar69a]:

If $A(s, t)$ satisfies Mandelstam analyticity and has positive double spectral functions, then the inelastic unitarity inequalities (17) imply the bound

$$0 \leq \text{Im } A(s+i0, 0) \leq \text{const } s (\log s)^{-1}, \quad s \geq 4. \quad (18)$$

Using additional consequences of the inequalities (14) Martin has also deduced that these amplitudes satisfy a Mandelstam representation of at most two subtractions.

Hence increasing total cross sections are excluded for unitary amplitudes with positive double spectral functions. The bound (18) is an optimal bound in the following respect. It can be saturated by crossing symmetric amplitudes, which have positive double spectral functions and satisfy the inelastic unitarity inequalities (14); see section 2.3 of [Kup71].

3.3 Gribov's Theorem

In 1960 Gribov [Gri60, Gri61] derived a consequence of crossing symmetry and elastic unitarity without any assumption about the partial waves at high energy. Let $A(s, t)$ be an amplitude,

which satisfies Mandelstam analyticity. Then elastic unitarity in the t -channel does not allow a linear increase of the absorptive part like

$$A_s(s, t) \simeq s f(t) \quad \text{if } s \rightarrow \infty \quad (19)$$

for t in some interval $t_1 < t < t_2$ with $t_1 < 0$ and $t_2 > 4$. Thereby $f(t)$ is a real analytic function with a cut starting at the elastic threshold $t = 4$. This statement is sometimes called *Gribov's Theorem*. With the same reasoning one can exclude an asymptotic behaviour $A_s(s, t) \simeq s^p (\log s)^q f(t)$ if $s \rightarrow \infty$ with any $p \in \mathbb{R}$ and $q \geq -1$, whereas values of $q < -1$ do not lead to a contradiction [Fro63]. Since the partial waves remain bounded only if $p < 1$ (and $q \in \mathbb{R}$) or $p = 1$ and $q \leq 0$, Gribov conjectured that the total cross section has to decrease for high energies.

In the language of Regge theory the result of Gribov means that the asymptotics cannot be dominated by a fixed pole at real angular momentum. One can circumvent the inconsistencies seen by Gribov in using rising complex Regge trajectories, see section 6. But there remain problems with the crossed channel contributions of the elastic unitarity integral, if one tries to combine elastic unitarity with the inelastic constraints (14); see section 6.2.

It should be stressed that Gribov's arguments are based on elastic unitarity and crossing. If one demands crossing symmetry and the inelastic unitarity inequalities (i. e. the constraints 1.– 3. and 5. of section 2), but not elastic unitarity, it is possible to find amplitudes which satisfy these constraints and which have the high energy behaviour (19). The construction of such amplitudes is shortly discussed in section 5.2.2.

4 Constructions using the Mandelstam representation

In this section we recapitulate the construction of amplitudes which satisfy crossing symmetry, elastic unitarity, and (partly) the inelastic unitarity constraints (14) using the Mandelstam representation explicitly. For these amplitudes elastic unitarity is obtained using a non-linear fixed point mapping for the spectral functions. This method has first been established by Atkinson for amplitudes which satisfy a Mandelstam representation without subtraction [Atk68a, Atk68b]; then it has been extended to amplitudes with one subtraction [Atk70, Kup69] and to amplitudes with an arbitrary (finite) number of subtractions [Atk69].

The results for amplitudes with positive spectral functions are recapitulated in section 4.1. The non-linear fixed point mapping for elastic unitarity is presented in section 4.2. The details are given only for amplitudes without subtraction. Amplitudes with an arbitrary number of subtractions are shortly discussed in section 4.3.

4.1 Amplitudes with positive spectral functions

Until now the proof of the existence of amplitudes, which satisfy all the requirements of Mandelstam analyticity, crossing symmetry, exact elastic unitarity, and the inelastic inequalities (14), has been given only for amplitudes, which satisfy a Mandelstam representation with at most one subtraction and which have positive spectral functions. The first proof has been given by Atkinson for amplitudes, which satisfy an unsubtracted Mandelstam representation $A(s, t) = \text{Sym } F(s, t)$ with

$$F(s, t) = \Phi_0[\psi](s, t) := \frac{1}{\pi^2} \int_4^\infty \int_{16}^\infty \psi(x, y) \frac{1}{(x-s)(y-t)} dx dy, \quad (20)$$

where $\psi(x, y)$ is a positive Hölder continuous function with support in $[4, \infty) \times [16, \infty)$. The proof is based on the construction of a non-linear fixed point equation for the double spectral function

such that any fixed point solution of this mapping satisfies exactly elastic unitarity. The existence of such a fixed point is established by the Leray-Schauder fixed point theorem [Atk68a], or by the contraction mapping theorem [Atk68b]. The validity of the inelastic constraints (14) follows from estimates of the partial waves of this fixed point solution. In [Atk68b] the generalization to charged pions is considered, and in [AW69] solutions with additional CDD ambiguities are constructed. For all these solutions the imaginary part of the forward amplitude vanishes like $\text{Im}A(s+i0,0) \lesssim s^\alpha(\log s)^{-\beta}$ with $-1 < \alpha < 0$ and $\beta < -1$.

In the next step a class of amplitudes $A(s,t) = \text{Sym} F(s,t)$ has been constructed which satisfy the once subtracted Mandelstam representation $F(s,t) = \Phi_1[\varphi,\psi](s,t)$ with

$$\Phi_1[\varphi,\psi](s,t) := \frac{1}{2\pi} \int_4^\infty \varphi(x) \frac{1}{x-s} dx + \frac{s+t}{\pi^2} \int_4^\infty \int_{16}^\infty \frac{\psi(x,y)}{x+y} \frac{1}{(x-s)(y-t)} dx dy \quad (21)$$

where $\varphi(x)$ and $\psi(x,y)$ are positive spectral functions. Elastic unitarity is again established by a fixed point equation for the spectral functions. The inelastic constraints (14) are obtained by explicit estimates for the partial wave amplitudes. The calculations for these estimates are indicated in Appendix A. The imaginary part of the forward amplitude can grow like

$$\text{Im} A(s+i0,0) \sim s^\alpha(\log s)^{-\delta} \quad \text{if } s \rightarrow \infty \quad (22)$$

with $0 < \alpha < 1$ and $\delta > 1$, or $\alpha = 1$ and $\delta \geq 3$. The amplitudes with $0 < \alpha < 1$ have been constructed in [Kup69] for $\delta = 2$; but solutions for any $\delta > 1$ exist. The solutions with $\alpha = 1$ have been obtained by Atkinson [Atk70]². The limitation to $\delta \geq 3$ is discussed below.

The non-linear mapping for elastic unitarity will be considered in more detail in section 4.2. Here we would like to add some comments about the optimal increase of the amplitudes. Knowing the bound (18) and the arguments of Gribov [Gri60, Gri61] one might expect that solutions exist, which satisfy elastic unitarity together with the inelastic bounds and increase like $\text{Im}A(s+i0,0) \sim s(\log s)^{-\delta}$ where δ can be a number just above 1. The reason for the restriction to $\delta \geq 3$ needs therefore some explanation. In [Atk70] the function $\psi(s,t) \geq 0$ is chosen such that

$$\psi(s,t) \sim \frac{t}{s}(\log t)^{-\delta}(\log s)^{-\lambda} \quad (23)$$

with $\delta > 1$ and $\lambda > 1$. This ansatz is motivated by Martin's paper [Mar69a] on amplitudes with positive double spectral functions. Assuming a smooth behaviour of the spectral functions (Hölder continuity) the absorptive part in the t -channel is bounded by $|A_t(s,t)| \leq \text{const} \cdot t(\log t)^{-\delta}$ for $t \rightarrow \infty$, and the bound (23) is stable under the iteration for elastic unitarity, see [Atk70]. The imaginary parts of the partial waves behave like $\text{Im} a_l(s) \sim (\log s)^{-\delta+1}$ and the real parts are bounded by $(\log s)^{-\delta+2}$ for high energies; see Appendix A.2. With these estimates the inelastic inequalities can be satisfied only if $\delta \geq 3$; and $\text{Im}A(s,0) \sim s(\log s)^{-3}$ is the strongest increase at high energies that can be derived with these constructions.

The weak part of the arguments leading to this conclusion is the estimate for the real parts of the partial waves. If we only demand the inelastic constraints (14) for $s \geq 4$, then crossing symmetric amplitudes with positive double spectral function have been constructed with an improved estimate of the partial waves, $|a_l(s)| \lesssim c \cdot (\log s)^{-\delta+1}$, where any $\delta \geq 1$ is admitted. Thereby the constraints 1. – 3. and 5. of section 2 can be satisfied such that the bound (18) of the forward amplitude $\text{Im} A(s+i0,0) \sim s(\log s)^{-1}$ is saturated, see section 2.3 of Ref. [Kup71]. To obtain the improved estimates for $\text{Re} a_l(s)$ cancellations between the s -channel and the u -channel contributions to $A_t(s,t)$ have to be taken into account. That is possible with an explicit ansatz (as done in [Kup71]). But if elastic unitarity is incorporated, the norm estimates

²The relation (1.1) in [Atk70] should read $\sigma(s) \approx (\log s)^{-3-\varepsilon}$ with $\varepsilon \geq 0$.

used for the fixed point mapping are not so precise to extract these cancellations. Nevertheless there remains a chance that the unitarity mapping can be treated with modified norms to reach values $\delta < 3$. But the restriction to $\delta > 1$ will remain as a consequence of the bound (18) and of Gribov's arguments.

Remark 1 *One can allow small negative contributions to the spectral functions. But the estimates have to be dominated by the positive parts. Especially, all spectral functions have to be positive near the boundary of their support. For the double spectral functions that statement is true independent of any specific construction: Mahoux and Martin [MM64] have derived domains where the double spectral functions have to be positive assuming elastic unitarity, crossing symmetry and the positivity of $\text{Im } a_l(s)$ for $s \geq 4$ and $l = 0, 1, 2, \dots$*

4.2 The fixed point mapping for elastic unitarity

In the case of Mandelstam analyticity elastic unitarity implies the following identity for the double spectral function [Man58]

$$\rho(s, t) = \int_4^\infty dt_1 \int_4^\infty dt_2 K(s, t, t_1, t_2) A_t(s + i0, t_1) A_t(s - i0, t_2), \quad (24)$$

valid in the domain $4 \leq s \leq 16$ and $t \geq 4$. The function K is the Mandelstam kernel

$$K(s, t, t_1, t_2) = \frac{2}{\pi \sqrt{s(s-4)}} \left(t^2 + t_1^2 + t_2^2 - 2(tt_1 + tt_2 + t_1t_2) - 4 \frac{tt_1t_2}{s-4} \right)_+^{-\frac{1}{2}}, \quad (25)$$

The support properties of this kernel imply that $\rho(s, t)$ has non-vanishing contributions only for

$$s > 4, t > 16 + \frac{64}{s-4} \quad \text{or} \quad t > 16, s > 4 + \frac{64}{t-16}. \quad (26)$$

For amplitudes (20) without subtraction the identity (24) is equivalent to elastic unitarity; in the case of n subtractions one needs also the identities (13) for the partial wave amplitudes with $0 \leq l < n$ to determine the single spectral functions.

To obtain a well defined fixed point problem the space of double spectral functions is equipped with a Banach space topology. In most publications spaces of functions $f(s, t)$, which are Hölder continuous in both variables have been used [Atk68a, Atk68b, AW69, Atk70, AFJK76, Kup69]. But it is also possible to work with an integral norm [Kup70b, Kup77]. This integral norm is more adequate for amplitudes with Regge poles, see section 6.

The space $\mathcal{L}_\gamma, \gamma > -\frac{1}{2}$, is the Hilbert space of all complex functions $\mathbb{R}_+ \ni t \rightarrow f(t) \in \mathbb{C}$ which have a finite norm

$$\|f\|_\gamma = \left[\int_0^\infty |t^{-\gamma} f(t)|^2 \left(1 + |\log t|^2 \right) \frac{dt}{t} \right]^{\frac{1}{2}}. \quad (27)$$

Functions $f(s, t)$ of two variables $s \geq 4$ and $t \geq 0$ are defined as Hölder continuous mappings $[4, \infty) \ni s \rightarrow f(s, \cdot) \in \mathcal{L}_\gamma$. More explicitly, a family of Banach spaces $\mathcal{L}(\gamma, \delta), \gamma > -\frac{1}{2}, \delta \in \mathbb{R}$, is introduced with the norms

$$\|f(s, t)\|_{\gamma, \delta} = \sup_{s \geq 4, 0 < h \leq 1} s^{-\delta} \left(\|f(s, t)\|_\gamma + h^{-\mu} \|f(s+h, t) - f(s, t)\|_\gamma \right). \quad (28)$$

Thereby μ is a Hölder index from the interval $0 < \mu < \frac{1}{2}$. Let $f(s, t)$ be a function with support in $s \geq 4$ and $t \geq 4$, then the function $\check{f}(s, t) := f(t, s)$ is obtained by an interchange

of the variables. For spectral functions $\rho \in \mathcal{L}(\gamma, \delta)$ with $\gamma < 0$ and $\delta < 0$ the double dispersion integrals (21) $\Phi_0[\rho]$ and $\Phi_0[\bar{\rho}]$ are well defined.

We consider the case without subtraction in some detail. Let $\psi(s, t) \in \mathcal{L}(\gamma, \gamma)$, $-\frac{1}{2} + \mu < \gamma < 0$, be a real double spectral function, which has a support in the (slightly extended) elastic domain $4 \leq s \leq 16 + \varepsilon$, $\varepsilon > 0$, and $16 \leq t < \infty$, and let $\omega(s, t) \in \mathcal{L}(\gamma, \gamma)$ be a real double spectral function, which has a support only in the inelastic region $s \geq 16$ and $t \geq 16$. Then the total amplitude is defined as

$$A(s, t) = E(s, t) + B(s, t) \quad \text{with} \quad (29)$$

$$E(s, t) := \text{Sym} \Phi_0[\psi](s, t) \quad \text{and} \quad B(s, t) := \text{Sym} \Phi_0[\omega](s, t). \quad (30)$$

The double spectral function of A is $\rho(s, t) = \psi(s, t) + \psi(t, s) + \omega(s, t) + \omega(t, s)$. As a consequence of the support restrictions for the spectral functions we have $\rho(s, t) = \psi(s, t)$ if $4 \leq s \leq 16$. Hence, if the identity

$$\psi(s, t) = \lambda(s) \int_4^\infty dt_1 \int_4^\infty dt_2 K(s, t, t_1, t_2) A_t(s + i0, t_1) A_t(s - i0, t_2) \quad (31)$$

is true for $s \geq 4$, the amplitude satisfies elastic unitarity. Thereby $\lambda(s)$ is a differentiable function with the properties $0 \leq \lambda(s) \leq 1$ if $s \in \mathbb{R}$, and

$$\lambda(s) = \begin{cases} 1 & \text{if } 4 \leq s \leq 16, \\ 0 & \text{if } s \geq 18. \end{cases} \quad (32)$$

This function cuts off the support of $\psi(s, t)$ at $s = 18$.

The fixed point mapping is now defined for the double spectral function $\psi(s, t)$ as follows. We fix a Hölder index μ with $0 < \mu < \frac{1}{2}$ and real parameters γ and δ with the constraints $-\frac{1}{2} + \mu < \gamma < \delta < 0$. Then a background contribution $B(s, t)$ is specified that has a double spectral function $\omega \in \mathcal{L}(\gamma, \gamma)$ with support in the inelastic region $s \geq 16$ and $t \geq 16$. Choosing a double spectral function $\psi(s, t) \in \mathcal{L}(\gamma, \gamma)$ with support in the domain $\{(s, t) \mid 4 \leq s \leq 18, t \geq 16\}$, the absorptive part $A_t(s, t) = \text{Abs}_t A(s, t)$ of the amplitude (29) is calculated as element of $\mathcal{L}(\gamma, \delta)$

$$A_t(s, t) = D[\psi](s, t) + B_t(s, t) \quad \text{with} \quad (33)$$

$$D[\psi](s, t) := \text{Abs}_t E(s, t) = \frac{1}{\pi} \int_4^\infty (\psi(x, t) + \psi(t, x)) \left(\frac{1}{x - s} + \frac{1}{x - 4 + s + t} \right) dx. \quad (34)$$

In (34) the integral $\pi^{-1} \int_4^\infty \psi(x, t)(x - s)^{-1} dx$ is the t -channel absorptive part of the function $\Phi_0[\psi](s, t)$; the other contributions arise from crossing. Then the unitarity integral (24) determines an image function

$$\psi'(s, t) = \lambda(s) \int_4^\infty dt_1 \int_4^\infty dt_2 K(s, t, t_1, t_2) A_t(s + i0, t_1) A_t(s - i0, t_2) \quad (35)$$

which is again an element of $\mathcal{L}(\gamma, \gamma)$ with a support in $4 \leq s \leq 18$ and $16 + 64(s - 4)^{-1} \leq t < \infty$. The equations (33) – (35) define a non-linear mapping $\psi \rightarrow \psi'$ for the real spectral function $\psi \in \mathcal{L}(\gamma, \gamma)$. This mapping is denoted as $\psi' = \Upsilon(\psi)$. Any fixed point of this mapping yields an amplitude (29), which satisfies crossing symmetry and elastic unitarity. Moreover, the fixed point solution is Hölder continuous in both variables [Kup70b] though we have used an integral norm. Therefore the assumption of Hölder continuity – as done in section 2 – is a natural one. A more detailed analysis shows that $\Upsilon(\psi)$ is a contraction mapping inside some ball $\|\psi\| \leq c$ provided the inhomogeneous contribution ω has a sufficiently small norm. Hence a unique fixed

point solution can be obtained by iteration. If $\omega \neq 0$ the trivial solution $\psi = 0$ is excluded. Moreover, if $\omega(s, t)$ is a positive function with dominant contributions near the boundary of its support, the estimates of Appendix A.2 can be used to derive that the fixed point solution satisfies the inelastic unitarity constraints (14) for $s \geq 16$.

Remark 2 *The conditions for the Banach contraction principle (or for the Leray-Schauder fixed point theorem) are only established for amplitudes with a sufficiently small norm. Hence this method does not lead to amplitudes with resonances. But a generalization to amplitudes with CDD ambiguities is possible [AW69].*

4.3 Amplitudes with an arbitrary number of subtractions

The fixed point mapping Υ can be generalized to a mapping for amplitudes with an arbitrary finite number of subtractions and for isospin-1 pions, see [Atk69]. This mapping is defined on the Cartesian products of the Banach spaces of the independent spectral functions, and it is possible to satisfy the conditions for a contraction mapping. Hence amplitudes exist, which satisfy the constraints 1.– 4. of section 2 and have a polynomial increase for large s and t of arbitrary strength. In the case of one subtraction the inelastic unitarity constraints can be incorporated for all energies as already stated in section 4.1, but in the case of more subtractions the partial waves are in general not bounded, and the forward amplitude may increase polynomially.

5 Regge type amplitudes that satisfy inelastic unitarity constraints

In this section we review the construction of amplitudes that satisfy the requirements 1.- 3. and 5. (together with 6.) of section 2 – only elastic unitarity is missing. These amplitudes can produce a constant or increasing total cross section. The construction starts from an explicit ansatz with a Regge type asymptotics. The investigations include models with simple and with double Regge poles [Kup71] and models with crossing Regge cuts [Kup82] that saturate the Froissart bound.

In the case of a leading Regge pole at angular momentum $l = \alpha(t)$ the asymptotic behaviour of the absorptive part is

$$A_s(s, t) \simeq \beta(t) s^{\alpha(t)} \text{ for } s \rightarrow \infty \quad (36)$$

where t lies in some interval, which includes $t = 0$. The Regge trajectory $\alpha(t)$ and the residue function $\beta(t)$ are real analytic functions with cuts starting at $t = 4$. If the amplitude has a double pole at angular momentum $l = \alpha(t)$ the amplitude has the asymptotic behaviour

$$A_s(s, t) \simeq \beta(t) s^{\alpha(t)} \log s \text{ for } s \rightarrow \infty. \quad (37)$$

The Froissart bound imposes the restriction $\alpha(0) \leq 1$ for the intercept. In the subsequent section 5.2 we recapitulate the construction of such Regge amplitudes. Thereby the limit case $\alpha(0) = 1$ is included.

In [Kup82] these methods have been extended to amplitudes with crossing Regge cuts. The main step for the construction of such amplitudes that saturate the Froissart bound is recapitulated in section 5.3.

Since the double spectral functions of Regge amplitudes are oscillating and increasing like $s^{\text{Re } \alpha(t)}$ for $s \rightarrow \infty$, the methods of section 4.1 and of Appendix A are no longer sufficient to derive the unitarity inequalities (14). As new technique a linearization of the quadratic inequalities is used. This method has been developed in [Kup71, Kup82] and it is presented in the next section before the details of Regge amplitudes are discussed.

5.1 The linear unitarity inequalities

Let $F(s, t)$ be an analytic function, which has partial wave amplitudes $f_l(s)$ that satisfy the relations

$$|f_l(s)| \leq c_1, \quad (38)$$

$$|\operatorname{Re} f_l(s)| \leq c_2 \operatorname{Im} f_l(s) \quad (39)$$

for $l = 0, 1, 2, \dots$ and $s \geq s_1 \geq 4$ with some constants $c_1 > 0$ and $c_2 \geq 0$. Then the inequalities $(\operatorname{Im} f_l)^2 \leq c_1 \operatorname{Im} f_l$, and $(\operatorname{Re} f_l)^2 \leq c_1 |\operatorname{Re} f_l(s)| \leq c_1 c_2 \operatorname{Im} f_l$ follow with the final result

$$|f_l|^2 \leq c_1(1 + c_2) \operatorname{Im} f_l, \quad l = 0, 1, 2, \dots, s \geq s_1 \geq 4. \quad (40)$$

After an appropriate scaling the partial wave amplitudes of $A(s, t) = c \cdot F(s, t)$ with $0 < c \leq (c_1 + c_1 c_2)^{-1}$ satisfy the normalized inelastic unitarity inequalities (14). Hence the constraints (38) and (39) imply the quadratic unitarity inequalities (14) for a scaled amplitude. The constraints (38) and (39) will be denoted as *linear unitarity inequalities*.

For the Regge amplitudes of this section the inelastic inequalities are derived in the linear form (38) and (39). Thereby the uniform bound (38) is obtained by a simple estimate of the integral (11). The essential tool to derive the inequalities (39) is an order relation between real functions [Kup71, Kup82].

We first define a set of functions $\phi(s, t)$ which have the following properties:

- $\alpha)$ The function $\phi(s, t)$ is Hölder continuous in $s \geq s_1 \geq 4$ and holomorphic in the cut plane $t \in \mathbb{C}_{cut}$, it is real if $s \geq s_1$ and $-s < t < 4$.
- $\beta)$ The partial wave amplitudes of $\phi(s, t) = 2\sqrt{\frac{s}{s-4}} \sum_l (2l+1) f_l(s) P_l\left(1 + \frac{2t}{s-4}\right)$ are positive, $f_l(s) \geq 0$, for $l = 0, 1, 2, \dots$, and $s \geq s_1$.

The set of functions with these properties is denoted by \mathcal{A} , or more precisely by $\mathcal{A}(s_1)$. If $\phi_1(s, t) \in \mathcal{A}$ and $\phi_2(s, t) \in \mathcal{A}$, then the sum $\alpha\phi_1(s, t) + \beta\phi_2(s, t)$ with $\alpha \geq 0$, $\beta \geq 0$, and the product $\phi_1(s, t)\phi_2(s, t)$ are also elements of \mathcal{A} . As a consequence of these simple statements more complicated constructions are possible; e.g., if $\phi(s, t) \in \mathcal{A}$, then also $\exp \lambda\phi(s, t) \in \mathcal{A}$ if $\lambda \geq 0$. For more details see Appendix B.1. The statement $\phi(s, t) \in \mathcal{A}(4)$ is the usual positivity constraint for the absorptive part of a scattering amplitude [Mar69b].

For functions which satisfy the conditions $\alpha)$ a partial order $\phi_1 \prec \phi_2$ can be defined by

$$\phi_1(s, t) \prec \phi_2(s, t) \quad \text{if} \quad \phi_2(s, t) - \phi_1(s, t) \in \mathcal{A}. \quad (41)$$

This relation is preserved by multiplication with a function $\chi(s, t) \in \mathcal{A}$

$$\phi_1 \prec \phi_2 \quad \leadsto \quad \chi \cdot \phi_1 \prec \chi \cdot \phi_2. \quad (42)$$

The unitarity inequalities (39) are equivalent to the linear relations

$$-c_2 \operatorname{Im} F(s + i0, t) \prec \operatorname{Re} F(s + i0, t) \prec c_2 \operatorname{Im} F(s + i0, t) \quad (43)$$

with $s \geq s_1 \geq 4$. To prove that the partial wave amplitudes of an analytic function $F(s, t)$ satisfy the inequalities (40) it is therefore sufficient to derive the uniform bounds (38) and to establish the order relations (43) with a number $c_2 \geq 0$.

The linear relations (38) and (39) are stronger than the quadratic inequalities, and they are not compatible with the double spectral region imposed by elastic unitarity.

5.2 Amplitudes with Regge poles

In this section we recapitulate the construction of crossing symmetric amplitudes, which satisfy crossing symmetry and the inelastic unitarity inequalities (14), and which exhibit the Regge asymptotics (36) or (37). All results have been derived for bounded Regge trajectories $\alpha(t)$. Moreover the imaginary part is assumed to be positive, $\text{Im } \alpha(t + i0) \geq 0$, such that $\alpha(t)$ is a convex increasing function for $t \in (-\infty, 4]$. The limit case of a fixed pole is admitted. From general arguments we know that the following restrictions must be obeyed: $\alpha(0) \leq 1$ (Froissart bound), and $\alpha(4) < 2$ [JM64]. Amplitudes with the following types of trajectories and residue functions have been constructed and will be discussed in the sequel:

1. The leading Regge singularity is a moving pole at $\alpha(t)$. The function $\alpha(t)$ is a bounded real analytic function with a cut starting at $t = 4$

$$\alpha(t) = \alpha(\infty) + \frac{1}{\pi} \int_4^\infty \frac{\text{Im } \alpha(x + i0)}{x - t} dx \quad (44)$$

The imaginary part $\text{Im } \alpha(t + i0)$ is Hölder continuous with support in the interval $4 \leq t < \infty$, it is strictly positive for $t > 4$. The intercept $\alpha(0)$ has a value in the interval $0 < \alpha(0) \leq 1$, and the limit $\alpha(\infty) = \lim_{t \rightarrow \infty} \alpha(t)$ can be any number $\alpha(\infty) < \alpha(0)$.

2. The leading Regge singularity is a fixed pole at $\alpha(t) = \alpha_0$, $t \in \mathbb{C}$, with $0 < \alpha_0 \leq 1$. Such amplitudes do not contradict Gribov's Theorem since elastic unitarity is not satisfied. A construction of the fixed pole solutions is indicated in section 5.2.2. Martin and Richard [MR01] have presented another simpler construction of an amplitude with a fixed pole at $\alpha(t) = 1$, $t \in \mathbb{C}$. But their proof of the inelastic inequalities does not include all partial wave amplitudes.
3. The Regge singularity is a double pole on a rising trajectory $\alpha(t)$ which has an intercept $\alpha(0) \leq 1$. The intercept can have the value $\alpha(0) = 1$ such that the total cross section (9) increases logarithmically. Moreover, there exist solutions with a fixed double pole at $l = \alpha < 1$.
4. The residue $\beta(t)$ is a real analytic function with a cut starting at $t = 4$. The partial wave coefficients of $\beta(t)$ are positive, and the function $\beta(t)$ has the upper bound $|\beta(t)| \leq \text{const } (1 + |t|)^{-\delta}$, $t \in \mathbb{C}_{\text{cut}}$, with $\delta > 2^{-1} (1 + \alpha(\infty))$.

5.2.1 Amplitudes with rising Regge trajectories

The construction of amplitudes with Regge poles follows [Kup71]. The amplitudes have the structure

$$A(s, t) = \hat{R}(s, u) + \hat{R}(s, t) + \hat{R}(t, u) + G(s, t). \quad (45)$$

Thereby $\hat{R}(s, u)$ is an $s - u$ symmetric Regge ansatz and $G(s, t)$ is an $s - t - u$ crossing symmetric background function. The construction proceeds in three steps:

1. In the first step an explicit ansatz for an $s - u$ symmetric Regge function $\hat{R}(s, u)$ is given with the following properties:
 - $\hat{R}(s, u)$ has the asymptotic behaviour (36),
 - the crossed amplitudes $\hat{R}(s, t)$ and $\hat{R}(t, u)$ vanish for $s \rightarrow \infty$ and t fixed.
2. The second step is the proof that $\hat{R}(s, u)$ satisfies (up to a scale factor) the inequalities (14) for energies $s \geq 20$. For that purpose the linear unitarity inequalities of section 5.1 are used.

3. The remaining task is the proof that one can find a crossing symmetric amplitude $G(s, t)$, which satisfies a Mandelstam representation with positive spectral functions (as discussed in section 4.1) such that (45) fulfills the unitarity inequalities (14) above threshold $s \geq 4$. If the trajectory function approaches negative values $\alpha(\infty) < 0$ for large momentum transfers, the background function $G(s, t)$ can be chosen to satisfy an unsubtracted Mandelstam representation.

Thereby it is possible to keep the threshold behaviour (15) in all steps of the construction.

To formulate an ansatz for $\hat{R}(s, u)$ a positive weight function $\sigma(s)$ is introduced with support in the interval $17 \leq s \leq 18$. This weight function is normalized to $\int \sigma(s) ds = 1$. The ansatz for $\hat{R}(s, u)$ is

$$\hat{R}(s, u) = -\frac{\beta(t)}{\sin \pi \gamma(t)} \int_{17}^{18} ds' \sigma(s') \int_{17}^{18} du' \sigma(u') [(s' - s)(u' - u)]^{\gamma(t)}, \quad (46)$$

where we have used $\gamma(t) := 2^{-1}\alpha(t)$ to simplify the notations. The absorptive part of $\hat{R}(s, u)$ behaves like (36) for $s \rightarrow \infty$ and t fixed. The function $\sigma(s)$ should be sufficiently differentiable such that the convolution with $\sigma(s)$ yields a smooth behaviour of $\hat{R}(s, u)$ at the thresholds of the variables s and u . The thresholds in s and u have been pushed to $s = 17$ ($u = 17$) such that (46) does not contribute to the double spectral function in the elastic domain $s \leq 16$. But it is possible to take any other threshold $s > 4$ for this construction. For large real values of s and fixed t the amplitude behaves like

$$\hat{R}(s + i0, u) \simeq -\frac{\beta(t)}{\sin \pi \gamma(t)} \exp(-i\pi \gamma(t)) s^{\alpha(t)} \quad \text{if } s \rightarrow \infty. \quad (47)$$

It is possible to choose a residue function $\beta(t)$ such that the crossed terms $\hat{R}(s, t) + \hat{R}(t, u)$ do not contribute to the asymptotics for large s . If $\alpha(\infty) < 0$ (and consequently $\gamma(\infty) < 0$) the factor $(\sin \pi \gamma(t))^{-1}$ has poles in the physical region $t < 0$. These poles have to be canceled by zeroes of the residue function $\beta(t)$.

For a Regge trajectory with $\alpha(0) \leq 1$ and $\alpha'(0) \geq 0$ the integral (11) yields a uniform bound (38) for the partial wave amplitudes

$$|f_l(s)| \leq c \sqrt{\frac{s-4}{s}} s^{\alpha(0)-1} \leq c \sqrt{\frac{s-4}{s}}, \quad l = 0, 1, 2, \dots, \quad s \geq 4. \quad (48)$$

For energies $s \geq 18$ and $4 - s \leq t \leq 0$ the imaginary and the real part of (46) $\hat{R}(s + i0, u)$ are given by

$$\text{Im } \hat{R}(s + i0, u) = \beta(t) \cdot \int ds' \sigma(s') (s - s')^{\gamma(t)} \int du' \sigma(u') [(u' - 4 + s + t)]^{\gamma(t)}, \quad (49)$$

$$\begin{aligned} \text{Re } \hat{R}(s + i0, u) &= -\beta(t) \cdot \cot \pi \gamma(t) \cdot \int ds' \sigma(s') (s - s')^{\gamma(t)} \\ &\quad \times \int du' \sigma(u') (u' - 4 + s + t)^{\gamma(t)}. \end{aligned} \quad (50)$$

Under appropriate assumptions about the residue function $\beta(t)$ it is possible to derive the linear unitarity relations (43) for these functions if $s \geq 20$. Hence the Regge ansatz $\hat{R}(s, u)$ satisfies the unitarity inequalities (40) for $s \geq 20$. The extension to the crossing symmetric amplitude (45) with the correct unitarity inequalities (14) for $s \geq 4$ follows the constructions in section 3.3 of [Kup71]. Some of the relevant calculations are given in the Appendices B.2 and B.3.

We would like to add three remarks.

Remark 3 To derive the inelastic unitarity inequalities for a sum like (45) the following statement is important. The set of amplitudes, which satisfy the inelastic unitarity conditions (14), is a convex cone: If $A_1(s, t)$ and $A_2(s, t)$ are two analytic functions with partial wave amplitudes, which fulfil the relations (14), then the sum $\alpha_1 A_1(s, t) + \alpha_2 A_2(s, t)$ has the same property, if the constants $\alpha_{1,2}$ are positive and $\alpha_1 + \alpha_2 \leq 1$.

Remark 4 The Regge ansatz (46) has a rectangular support of its double spectral region $\{(s, t) \mid 17 \leq s < \infty, 4 \leq t < \infty\}$. Such a support is compatible with the linear unitarity inequalities (39). It is possible to choose the background term $G(s, t)$ such that the full amplitude (45) has a double spectral region as required by elastic unitarity.

Remark 5 There are two essential ingredients for the construction of Regge amplitudes presented in this section: The imaginary part of the trajectory function $\alpha(t)$ is positive, and the residue $\beta(t)$ function has positive partial wave coefficients. These constraints are not compatible with elastic unitarity, see section 6.2.

5.2.2 Amplitudes with a fixed pole

The construction for amplitudes with a fixed pole at angular momentum $l = \alpha$ with $0 < \alpha \leq 1$ is the same as for amplitudes with rising trajectories. We start again from the ansatz (46). Since $\cot \pi \gamma$ is a (finite) number the prove of the linear relations (43) for $\hat{R}(s + i0, u)$ is simpler. The partial wave amplitudes have again the uniform bound (48). In the particularly interesting case $\alpha = 1$ ($\gamma = \frac{1}{2}$) the real part of the amplitude $\hat{R}(s + i0, u)$ vanishes for $s \geq 18$, and the relations (43) follow immediately for these energies. The extension to a crossing symmetric amplitude (45) $A(s, t)$ is done as in the case of rising Regge trajectories.

5.2.3 Amplitudes with double poles

The extension to amplitudes with the asymptotic behaviour (37) has been given in section 4. of [Kup71]. In the language of Regge poles the behaviour (37) corresponds to a double pole at angular momentum $l = \alpha(t)$. In the first step the ansatz (46) is generalized to an amplitude $\hat{R}_1(s, u)$ with a leading term $\sim s^{\alpha(t)} \cdot \log s$. Let $\sigma(x) \geq 0$, $x \in \mathbb{R}$, be a positive continuous function on the real line which has its support in the interval $17 \leq x \leq 18$ and which is normalized to $\int \sigma(x) dx = 1$. One can take the function $\sigma(x)$ used in (46). Then $L(s) := \int \sigma(x) \log(x - s) dx$ defines a smooth version of the logarithm on the cut plane \mathbb{C}_{cut} . The boundary values at $s = x \pm i0$, $x \geq 4$, are Hölder continuous with any index $0 < \mu < 1$. The $s - u$ symmetric function

$$H(s, t) = L(s) + L(4 - s - t) - L(t + 1) \quad (51)$$

is holomorphic in $(s, t) \in \mathbb{C}_{cut}^2$, and it has the upper bounds $|\text{Im } H(s, t)| \leq 3\pi$ and $|\text{Re } H(s, t)| \leq 2 \log(1 + |s|) + \text{const}$ for $(s, t) \in \mathbb{C}_{cut}^2$. If $s \geq 20$ and $-s < t < 4$ it can be written as

$$H(s \pm i0, t) = h(s, t) \mp i\pi, \quad (52)$$

with a positive real part $h(s, t) = \int \sigma(x) \log[(s - x)(x - 4 + s + t)(x - t - 1)^{-1}] dx > 0$. For large s the real part has the asymptotic behaviour $h(s, t) \simeq 2 \log s$. Moreover $h(s, t)$ has positive partial wave amplitudes, $h(s, t) \in \mathcal{A}(20)$, see Appendix B.1.

The Regge ansatz (46) is now modified to

$$\hat{R}_1(s, u) = \frac{1}{2} H(s, t) \cdot \hat{R}(s, u) \quad (53)$$

For $s \rightarrow \infty$ along the real line the absorptive part behaves like

$$\text{Abs}_s \hat{R}_1(s, u) \simeq \beta(t) s^{\alpha(t)} \log s. \quad (54)$$

The partial wave amplitudes of $\hat{R}_1(s, u)$ are uniformly bounded, if the trajectory has the properties $\alpha(0) < 1$ and $\alpha'(0) \geq 0$ or $\alpha(0) = 1$ and $\alpha'(0) > 0$. Therefore amplitudes with double poles can be constructed either if the pole position lies on a rising trajectory $\alpha(t)$ with an intercept $\alpha(0) \leq 1$, or if the pole has a fixed position at $l = \alpha < 1$.

The linear unitarity relations (43) are derived for $\hat{R}_1(s, u)$ with the techniques presented in section 5.1. The extension to a crossing symmetric amplitude

$$A(s, t) = \hat{R}_1(s, u) + \hat{R}_1(s, t) + \hat{R}_1(t, u) + G(s, t) \quad (55)$$

is done as for the amplitudes (45). The crossing symmetric function $G(s, t)$ is again a background term, which satisfies a Mandelstam representation without or with one subtraction. The asymptotic behaviour for $s \rightarrow \infty$ of the full amplitude $A(s, t)$ is dominated by $\hat{R}_1(s, u)$, and (54) implies (37). If $\alpha(0) = 1$ the forward amplitude behaves like $A(s, 0) \simeq \hat{R}_1(s, u = 4 - s) \simeq i\beta(0) s (\log s - i\frac{\pi}{2})$ for $s \rightarrow +\infty$

5.3 Amplitudes that saturate the Froissart bound

From an investigation of the forward peak of the scattering amplitude one knows that an amplitude, which saturates the Froissart bound and obeys the inelastic constraints must have a $\sqrt{-t} \log s$ shrinking of the forward peak [AKM71]. A behaviour of this type can be achieved by a Regge model with complex conjugate Regge trajectories (poles or cuts), which cross at $t = 0$, see e. g. [Ede71, Oeh72]. But this literature does not tell us, how crossing symmetry can be imposed [Khu76]. Fortunately, it is possible to follow the steps used for the Regge amplitudes in the last sections. In [Kup82] a class of amplitudes has been constructed that saturate the bound and that rigorously satisfy the requirements 1.– 3. and 5. of section 2. The essential part of this paper is the ansatz of an $s - u$ symmetric amplitude $\hat{F}(s, u)$ that satisfies

- the requirements 1. and 2. of section 2,
- the unitarity inequalities (14) (up to a scaling factor) beyond some energy $s \geq s_1 \geq 4$,
- the saturation of the Froissart bound.

The full crossing symmetric amplitude, which fulfills all constraints with exception of elastic unitarity, can then be obtained following the methods developed for the Regge amplitudes.

We indicate a few steps of the construction of $\hat{F}(s, u)$. Starting from the $s - u$ symmetric function (51) $H(s, t) = H(4 - s - t, t)$ we define the function

$$\Gamma(s, t, z) = -\frac{1}{\cos(\pi/2)z} \exp \left[\frac{1+z}{2} H(s, t) \right] \quad (56)$$

of the complex variables s, t and z . This function is holomorphic in the variables (s, t) in the Mandelstam domain \mathbb{C}_{cut}^2 , and it is holomorphic in the variable z in the open unit disc $\{z \mid |z| < 1\} \subset \mathbb{C}$ of the complex plane. The s -channel absorptive part is, see (52),

$$\text{Abs}_s \Gamma(s, t, z) = \left(\tan \frac{\pi}{2} z + i \right) \exp \left[\frac{1+z}{2} h(s, t) \right] \quad \text{if } s \geq 18. \quad (57)$$

Let $\gamma(\xi)$ be a real analytic function of the variable $\xi \in \mathbb{C} \setminus [2, \infty)$, vanishing at $\xi = 0$, $\gamma(0) = 0$, and with a Hölder continuous and positive imaginary part $\text{Im } \gamma(x + i0) \geq 0$ for $x \geq 2$. We assume a uniform bound

$$|\gamma(\xi)| \leq \delta < 1 \text{ for } \xi \in \mathbb{C} \setminus [2, \infty). \quad (58)$$

Then the function

$$\Phi(s, t; \gamma) := \frac{1}{\sqrt{t}} \left[\Gamma(s, t, \gamma(\sqrt{t})) - \Gamma(s, t, \gamma(-\sqrt{t})) \right] \quad (59)$$

is an $s - u$ symmetric amplitude, which is analytic in the Mandelstam domain. For $t < 0$ it develops two complex conjugate Regge trajectories of simple poles at angular momenta $l = 1 + \gamma(\pm i\sqrt{-t})$, which cross at $t = 0$. Moreover, the function $\Phi(s + i0, t; \gamma)$ satisfies the linear unitarity relations of section 5.1 beyond the energy $s = 20$. The proof for this statement is given in section 3 of [Kup82].

The real part of γ along the imaginary axis $\text{Re } \gamma(i\tau) = \text{Re } \gamma(-i\tau)$ is a monotonically decreasing function of $\tau \in [0, \infty)$ with values from $\gamma(0) = 0$ to $\gamma(\infty) < 0$. For values of $s > 18$ and $t < 0$ the imaginary part of (59) is given by

$$\text{Im } \Phi(s + i0, t; \gamma) = \frac{2}{\sqrt{-t}} \exp \frac{(1 + a(t)) h(s, t)}{2} \cdot \sin \left(\frac{b(t)}{2} h(s, t) \right)$$

with $a(t) = \text{Re } \gamma(i\sqrt{-t}) < 0$ and $b(t) = \text{Im } \gamma(i\sqrt{-t}) > 0$. The asymptotic behaviour is

$$\text{Im } \Phi(s + i0, t; \gamma) \simeq 2\beta_0(t) s^{1+a(t)} \frac{\sin [b(t) \log s]}{\sqrt{-t}}$$

with $\beta_0(t) = 2 \exp \left[-\frac{1}{2} L(t) (1 + a(t)) \right]$. At $t = 0$ we have

$$\text{Im } \Phi(s + i0, 0; \gamma) \sim s \log s \quad \text{if } s \rightarrow \infty \quad (60)$$

as for the amplitude (54) with the Regge double pole. But the amplitude $\Phi(s, t; \gamma)$ has – in contrast to the amplitudes of section 5.2 – a $\sqrt{-t} \log s$ shrinking of the forward peak. It satisfies the bound

$$|\Phi(s + i0, t; \gamma)| \leq c \cdot \chi_{1,1}(s, t) \quad \text{if } s \geq 4 \text{ and } 4 - s \leq t \leq 0 \quad (61)$$

with some positive constant c and the function

$$\chi_{m,n}(s, t) = \begin{cases} s(\log s)^m (1 + \sqrt{-t} \log s)^{-n} & \text{if } -1 \leq t \leq 0, \\ s^p (-t)^{-(1+p)/2} & \text{if } t < -1, \end{cases} \quad (62)$$

where p is a parameter in the interval $0 < p < 1$, and m and n are integers.

The amplitude $H(s, t)\Phi(s, t; \gamma)$ saturates the Froissart bound, but its partial wave amplitudes are not bounded for $s \rightarrow \infty$. It is necessary to improve the shrinking of the forward peak. For that purpose we introduce a smooth positive weight function $\rho(\lambda)$ which satisfies

$$\begin{aligned} \rho(\lambda) &\in \mathcal{C}^3(\mathbb{R}), \quad \rho(\lambda) \geq 0, \quad \int \rho(\lambda) d\lambda = 1, \\ \rho(\lambda) &= 0, \quad \text{if } \lambda \notin [\lambda_1, \lambda_2], \quad 0 < \lambda_1 < \lambda_2 < \delta^{-1}. \end{aligned} \quad (63)$$

If $\lambda \in [\lambda_1, \lambda_2]$ then $\lambda\gamma(\xi)$ has the same properties as demanded for $\gamma(\xi)$, the bound (58) is replaced by $|\lambda\gamma(\xi)| \leq \lambda_2 \delta < 1$. The integral

$$\tilde{\Phi}(s, t) = \int_{\lambda_1}^{\lambda_2} \rho(\lambda) \Phi(s, t; \lambda\gamma) d\lambda \quad (64)$$

defines again an analytic function in the Mandelstam domain. The λ -integration leads to a rapidly decreasing function of $\sqrt{-t} \log s$ yielding a stronger shrinking of the forward peak

$$\left| \tilde{\Phi}(s + i0, t) \right| \leq \text{const} \cdot \chi_{1,3}(s, t) \quad \text{if } s \geq 4 \text{ and } 4 - s \leq t \leq 0. \quad (65)$$

whereas the forward amplitude $\tilde{\Phi}(s, 0) = \Phi(s, 0; \gamma)$ remains unchanged.

Let $\beta(t)$ be a bounded real analytic function of the complex variable $t \in \mathbb{C} \setminus [4, \infty)$ with positive partial wave coefficients. Then the $s - u$ symmetric amplitude

$$\hat{F}(s, u) = \beta(t)H(s, t)\tilde{\Phi}(s, t) \quad (66)$$

saturates the Froissart bound. Since $\Phi(s + i0, t; \gamma)$ fulfils the linear unitarity relations (43), it is straightforward to derive such relations also for $\hat{F}(s, u)$ beyond the energy $s \geq 20$. The extension to a crossing symmetric amplitude

$$A(s, t) = \hat{F}(s, u) + \hat{F}(s, t) + \hat{F}(t, u) + G(s, t)$$

which satisfies the inelastic constraints for all energies $s \geq 4$ follows the same steps as used for the Regge amplitudes of section 5.2. The background function $G(s, t)$ can again be represented by a crossing symmetric Mandelstam integral without subtraction and with a positive double spectral function. For the details see [Kup82].

6 Regge amplitudes that satisfy elastic unitarity

In the first part of this section we discuss the construction of amplitudes, which satisfy

- Mandelstam analyticity and crossing symmetry in s, t and u ,
- elastic unitarity for $4 \leq s \leq 16$,
- the inelastic unitarity inequalities $|a_l(s)| \leq 1$ for $s \geq 16$,
- Regge asymptotics with a trajectory $\alpha(t)$, which has an intercept $0 < \alpha(0) \leq 1$.

If one does not care about unitarity constraints in the inelastic region, the trajectory function $\alpha(t)$ can be rather arbitrary, and an intercept $\alpha(0) > 1$ is possible. The remaining – and still unsolved problem – to prove the existence of a Regge amplitude, which satisfies crossing symmetry, elastic unitarity and all inelastic constraints, is discussed in section 6.2.

So far there is no extension of the fixed point equations of section 6.1 to amplitudes with double poles or to amplitudes with crossing Regge trajectories and cuts as needed for the saturation of the Froissart bound.

6.1 The fixed point problem for Regge amplitudes

It is possible to apply the fixed point method of section 4.2 to amplitudes, which have a Regge ansatz of the type presented in section 5.2.1 as inhomogeneous term, see e.g. [Kup70a]³. But then one has to assume that $\text{Re } \alpha(t) < 1$ for $t \leq 16$, and in the physical region the asymptotics is dominated by the background generated by the iteration. One therefore needs a more elaborate technique, which exposes the Regge pole explicitly. For that purpose three approaches have been used in the literature:

- the use of partial wave amplitudes and the Watson-Sommerfeld transform,
- the use of partial wave amplitudes and N/D equations, and
- the Khuri representation of the amplitude.

Elastic unitarity can be easily formulated for partial wave amplitudes with complex angular momenta. But the analyticity properties of the Watson-Sommerfeld transform cause some problems in constructing a crossing symmetric scattering amplitude with the correct Mandelstam analyticity. These problems have been solved by Atkinson and his collaborators [AFJK76, Fre75].

³The Regge amplitudes constructed in section 2.1 of [Kup70a] do not have all the properties indicated there. But one can substitute these erroneous Regge amplitudes by the amplitudes defined in section 3. of [Kup71] (and in section 5.2.1 of the present paper).

But unfortunately the fixed point equation obtained in [AFJK76] does not guarantee elastic unitarity for the pole term.

The N/D equations used by Johnson and Warnock [JW77, JWK77, War81] have the advantage that the Regge pole appears as zero of the denominator function, and analyticity and unitarity of the pole term is naturally included in the equations. But these authors have not given a conclusive proof, whether and under what conditions a fixed point solutions exists.

The following account is based on [Kup77] and uses the Khuri representation [Khu63], in which the correct analyticity is easily exposed, but the unitarity identity is more involved. The calculations can also be performed with the Watson-Sommerfeld integral representation (borrowing methods from the publications [AFJK76, Fre75] to obtain the correct analyticity). Since elastic unitarity imposes a non-linear relation between the Regge term and the background term, the fixed point mapping has to refer to the degrees of freedom of the Regge trajectory and of the background amplitude.

In the subsequent arguments the Regge poles appear in the large t asymptotics. The trajectory is therefore written as function of s . We consider an amplitude with one Regge trajectory $\alpha(s)$, which has the following properties:

- a) The function $\alpha(s)$ is real analytic and bounded. It has a Hölder continuous and strictly positive imaginary part $\text{Im } \alpha(s + i0) > 0$ if $s > 4$.
- b) The threshold behaviour of the trajectory is $\text{Im } \alpha(s + i0) \simeq c(s - 4)_+^{\sigma + \frac{1}{2}}$, $s \approx 4$, with the exponent $\sigma = \alpha(4)$.
- c) The values of $\alpha(s)$ are restricted to $-1 < \alpha(\infty) < -\frac{1}{2}$, $0 < \alpha(4) < 2$, and $\text{Re } \alpha(s) < \gamma_1$ if $s \in \mathbb{C}_{cut}$. Thereby γ_1 is a constant in the open interval $2 < \gamma_1 < 3$.

The strictly positive imaginary part of $\alpha(s + i0)$ in the elastic interval $4 < s \leq 16$ is needed to circumvent the problem encountered by Gribov, see section 3.3. For the actual construction we assume the stronger constraints a). The constraint b) is a consequence of elastic unitarity [BZ62, CS63]. The bounds c) simplify the construction. Values of the intercept $\alpha(0) \approx 1$ are included. Since we do not yet demand the inelastic constraints, also solutions with $\alpha(0) > 1$ exist.

The trajectory function is represented by

$$\alpha(s) = \alpha[\chi](s) = \alpha_0(s) + \frac{s-4}{\pi} \int_4^{18} \sqrt{\frac{x-4}{x}} (x-4)^{\sigma-1} \frac{\chi(x)}{x-s} dx \quad (67)$$

The real analytic function $\alpha_0(s)$ is an input function for the calculations. It has a cut at $s \geq 16$ and a positive imaginary part. Since the value of $\alpha(4)$ determines the threshold behaviour of the trajectory, we choose the subtraction point in (67) at $s = 4$. The value of $\alpha_0(4) = \alpha(4) = \sigma$ is restricted to $0 < \sigma < 2$, and we assume $-1 < \alpha_0(\infty) < -\frac{1}{2}$. The trajectory (67) is a functional of the strictly positive Hölder continuous function $\chi(x)$, which is defined on the interval $4 \leq x \leq 16$ and extended to $x \geq 16$ by $\chi(x) = \chi(16)\lambda(x)$ with the cut off function (32).

The scattering amplitude has the structure (8)

$$A(s, t) = \text{Sym } F(s, t) \quad \text{with} \quad F(s, t) = R(s, t) + G(s, t). \quad (68)$$

The functions $R(s, t)$ and $G(s, t)$ are holomorphic in the domain (7). The term $R(s, t)$ has a Regge asymptotics $\text{Abs}_t R(s, t) \simeq \beta(s)t^{\alpha(s)}$ for $t \rightarrow \infty$. The function $G(s, t)$ is a background term. The ansatz (68) corresponds to a strip approximation [CJ64]. To treat the Regge poles we

use the Khuri representation with respect to the variable t . The function $F(s, t)$ is represented by the Mellin-Barnes integral

$$F(s, t) = -\frac{1}{2i} \int_{\mathcal{C}} \frac{f(s, \nu)}{\sin \pi \nu} (-t)^\nu d\nu. \quad (69)$$

The curve of integration \mathcal{C} goes from $v = \gamma_0 - i\infty$ to $v = \gamma_0 + i\infty$ with $-\frac{1}{2} < \gamma_0 < 0$, the (Khuri) poles of the meromorphic function $f(s, \nu)$ lie to the left side of the curve and the integers $\nu = 0, 1, 2, \dots$ lie to the right side. The function $f(s, \nu)$ is the Mellin transform of $F_t(s, t) = \text{Abs}_t F(s, t)$, and the absorptive part of $F(s, t)$ in the t -channel is given by, see Appendix C.1,

$$\text{Abs}_t F(s, t) = \frac{1}{2i} \int_{\gamma_1} f(s, \nu) t^\nu d\nu. \quad (70)$$

The symbol $\int_{\gamma} d\nu \dots$ means integration along the line $\{\nu = \gamma + ix \mid -\infty < x < \infty\}$ with the fixed real part $\text{Re } \nu = \gamma$. The line of integration in (70) has to satisfy $\gamma_1 > \sup \{\text{Re } \alpha(s) \mid s \in \mathbb{C}_{\text{cut}}\}$. In the case without Regge poles as considered in section 4.2 we have $F(s + i0, t + i0) = \Phi_0[\psi](s + i0, t + i0) \in \mathcal{L}(\gamma_0, \delta)$ with $-\frac{1}{2} + \mu < \gamma_0 < \delta < 0$, and the function $f(s, \nu)$ is holomorphic in ν in the half plane $\{\nu \mid \text{Re } \nu > \gamma_0\}$. Then the line of integration in (69) and (70) can be pushed back to $\text{Re } \nu = \gamma_0$. If a Regge pole at position $\nu = \alpha(s)$ enters the half plane $\{\nu \mid \text{Re } \nu > \gamma_0\}$ the function $f(s, \nu)$ has a series of Khuri poles at positions $\nu = \alpha(s), \alpha(s) - 1, \alpha(s) - 2, \dots$. In a simplified picture, which neglects the Khuri daughter poles (and the Hölder continuity in the variable t), the separation of $f(s, \nu)$ into Regge pole contribution and holomorphic background can be written as

$$f(s, \nu) = \frac{\beta(s)}{v - \alpha(s)} t_1^{\alpha(s) - \nu} + g(s, \nu). \quad (71)$$

A more adequate pole term is given in Appendix C.2. The Mellin transform $a(s, \nu) = \mathcal{M}[A_t(s, t)](\nu)$ of the full amplitude (68) is then

$$a(s, \nu) = f(s, \nu) + \text{crossed terms} = \frac{\beta(s)}{v - \alpha(s)} t_1^{\alpha(s) - \nu} + b(s, \nu). \quad (72)$$

Thereby the crossed Khuri pole terms contribute only to the holomorphic background $b(s, \nu)$.

The elastic unitarity equation (31) has been calculated for the Khuri representation in [Kup77]; a few details are also given here in Appendix C.3. In this section we only use the truncated form

$$w'(s, \nu) = \frac{1}{2} \lambda(s) \sqrt{\frac{s-4}{s}} (s-4)_+^\nu B(1+\nu, 1+\nu) a(s+i0, \nu) a(s-i0, \nu), \quad (73)$$

which exhibits the essential consequences of the exact identity (144). Thereby $B(x, y)$ is the Euler beta function. The function $w'(s, \nu) = \text{Abs}_s f'(s, \nu)$ is the Mellin transform of the double spectral function of the image amplitude $F'(s, t)$ generated by the fixed point mapping. Without Khuri poles we can use the dispersion integral

$$f'(s, \nu) = \frac{1}{\pi} \int_4^{18} w'(x, \nu) \frac{1}{x-s} dx \quad (74)$$

to calculate the Mellin transform of $\text{Abs}_t F'(s, t)$. But if $a(s, \nu)$ has a pole contribution (72), then in the general case the integral (74) does not produce a pole but a cut. One has to satisfy rather exceptional conditions to keep a Regge pole stable under the iteration. To see this condition we

assume that $w'(s, \nu) = w(s, \nu) = (2i)^{-1} (f(s + i0, \nu) - f(s - i0, \nu))$ is a solution of the equation (73). Comparing the residues at $\nu = \alpha(s + i0)$ on both sides of (73) we obtain the following identity in the elastic interval $4 \leq s \leq 16$

$$\bar{\beta} = \sqrt{\frac{s}{s-4}} (s-4)_+^{-\alpha} B^{-1} (1 + \alpha, 1 + \alpha) \text{Im } \alpha - 2i b(s - i0, \alpha) \text{Im } \alpha \quad (75)$$

with $\alpha = \alpha(s + i0)$ and $\bar{\beta} = \beta(s - i0) = \overline{\beta(s + i0)}$. This identity relates the analytic functions $\alpha(s)$, $\beta(s)$ and the background term $b(s, \nu)$ in a highly non-trivial manner. The ansatz (67) with $\text{Im } \alpha(s + i0) \sim (s - 4)^{\sigma + \frac{1}{2}}$ compensates exactly the singularity of the right hand side at the threshold $s = 4$. An essential point is: The exact form of the equation (73) leads also to an identity, which can be written in the form (75); only the interpretation of the holomorphic function $b(s, \nu)$ has changed. The same type of identity emerges, if one performs the calculations with partial wave amplitudes.

The fixed point mapping for elastic unitarity has to take into account the constraint (75). In [Kup77] a non-linear mapping $\chi \rightarrow \chi' = T[\chi, b]$ for the unknown function $\chi(x)$ in (67) was built up from the real and the imaginary parts of (75) and the Hilbert transforms between the imaginary and the real parts of $\alpha(s)$ and of $\beta(s)$. This mapping preserves the positivity of $\text{Im } \alpha(s + i0)$: if $\chi(s)$ is positive for $4 \leq s \leq 16$, the image $\chi'(s)$ is also a positive function. Given the background function $b(s, \nu)$ the fixed point solution $\chi = T[\chi, b]$ leads to analytic functions $\alpha(s)$ and $\beta(s)$, which satisfy the identity (75). Thereby a ghost killing factor $\alpha(s)$ can be inserted into the residue function $\beta(s)$ in order to cancel the pole of $R(s, t) + R(s, u)$ at angular momentum $l = 0$, see Appendix C.2. Moreover one can prescribe a strong decrease of $\beta(s)$ such that the crossed Regge terms contribute only to the background.

The background is determined by a mapping $b \rightarrow b' = \Upsilon[\chi, b]$, which incorporates elastic unitarity and crossing for the background if the Regge term is given. The full non-linear mapping is the Cartesian product $T \times \Upsilon$

$$\begin{pmatrix} \chi \\ b \end{pmatrix} \rightarrow \begin{pmatrix} \chi' \\ b' \end{pmatrix} = \begin{pmatrix} T[\chi, b] \\ \Upsilon[\chi, b] \end{pmatrix}. \quad (76)$$

A detailed norm estimate of all steps yields that the square of this mapping is a contraction provided some norm restrictions are satisfied. The fixed point solution can therefore be obtained by iteration.

In [Kup77] the additive function $G(s, t)$ in (68) is omitted, and the role of the inhomogeneous term is taken over by an additive (sufficiently small) constant $\tau > 0$ in the dispersion relation for $\beta(s)$. The fixed point solution depends continuously on this constant. The norm restrictions imply that χ and β are small, but they are strictly positive (at least for $4 \leq s \leq 16$) if $\tau > 0$. The shape of the trajectory $\alpha(s)$ is essentially determined by the function $\alpha_0(s)$, which can be chosen within the constraints given above. The fixed point solution of (76) depends continuously on $\alpha_0(s)$. The intercept of the trajectory is

$$\alpha(0) = \alpha_0(0) - \frac{4}{\pi} \int_4^{18} x^{-\frac{3}{2}} (x - 4)^{\sigma - \frac{1}{2}} \chi(x) dx < \alpha_0(0). \quad (77)$$

If we start with an input function $\alpha_0(s)$, which has an intercept $\alpha_0(0) = 1$, the intercept of α is smaller, $\alpha(0) < 1$. But we can start with a function $\alpha_0(s)$ which has an intercept $\alpha_0(0) = 1 + \delta$, $0 < \delta < 1$. Choosing a small parameter $\tau > 0$ fixed point solutions with an arbitrarily small $\sup_s \chi(s)$ exist. Hence there are solutions $\chi(s)$ such that $\alpha(0) > 1$. The existence of amplitudes (68) with a Regge trajectory $\alpha(s)$ which has exactly the intercept $\alpha(0) = 1$ is then

the consequence of the continuity of the fixed point solution with respect to the input function $\alpha_0(s)$.

If $\alpha(0) \leq 1$ the partial wave amplitudes of $R(s, t) + R(s, u)$ are uniformly bounded. One can achieve that the residue function $\beta(s)$ decreases fast enough such that crossed terms do not contribute to the leading asymptotic behaviour, and the partial wave amplitudes of the full amplitude (68) are also uniformly bounded. This bound is a continuous function of the norm of the amplitude. Since we can obtain solutions with arbitrarily small norm, solutions with partial wave amplitudes bounded by unity exist.

6.2 Both constraints: elastic and inelastic unitarity

The inelastic unitarity inequalities (14) include the weaker constraint of positivity:

- The amplitude $A(s, t)$ satisfies *positivity* if $\text{Im } a_l(s) \geq 0$ for $l = 0, 1, 2, \dots$ and all energies $s \geq 4$.

Up to now there is no conclusive argument that the fixed point equations of section 6.1 have solutions, which satisfy positivity. A fortiori there is no proof that the fixed point equations have solutions, which satisfy the full inelastic unitarity inequalities.

We give some details for a better understanding of this remaining problem. From Mahoux and Martin [MM64] we know a consequence of elastic unitarity: the double spectral function has to be positive in a (precisely defined) neighbourhood of the boundary of its support. This fact implies that the imaginary parts $\text{Im } a_l(s)$ are positive for sufficiently large angular momenta, $l \geq \Lambda(s)$, and energies $s > 20$. Thereby $\Lambda(s)$ is an unknown function with the range $0 \leq \Lambda(s) < \infty$.

One can add an inhomogeneous term $G(s, t)$ to the Regge term $R(s, t)$, see (68). This term should be given by an unsubtracted crossing symmetric Mandelstam integral with positive double spectral function. Borrowing some arguments from Appendix B.3 it is possible to obtain solutions, which satisfy crossing symmetry, elastic unitarity and positivity within some finite range of energy.

But the proof of the positivity for the partial wave amplitudes with $l < \Lambda(s)$ at higher energies remains open. The problem originates from the partial wave amplitudes of the crossed Regge term $R(t, s)$, which dominates the s -channel asymptotics, $\text{Abs}_s R(t, s) \simeq \beta(t) s^{\alpha(t)}$ if $s \rightarrow \infty$. Unfortunately one cannot use arguments from section 5.2. There the inelastic inequalities (14) have been derived under the condition that $\text{Im } \alpha(t + i0) \geq 0$ and that $\beta(t)$ has positive partial wave coefficients. But this property does not hold for solutions of equation (75). To see that we first derive

Lemma 1 Let $\alpha(t)$ and $\beta(t)$ be real analytic functions which satisfy the identity (75). If $\text{Im } \alpha(t + i0) = c_1(t - 4)^{\sigma + \frac{1}{2}} + \mathcal{O}\left((t - 4)^{\sigma + \frac{1}{2} + \mu}\right)$ for $t \rightarrow 4$ with $c_1 > 0$, $\sigma = \alpha(4) > 0$ and $\mu > 0$, then near threshold the imaginary part of $\beta(t + i0)$ has the form

$$\text{Im } \beta(t + i0) = c_2(t - 4)^{\sigma + \frac{1}{2}} \log(t - 4) + \mathcal{O}\left((t - 4)^{\sigma + \frac{1}{2}}\right), \quad (78)$$

where c_2 is a positive constant.

Proof The functions $\mathbb{R} \ni t \rightarrow \alpha(t + i0)$ and $\mathbb{R} \ni t \rightarrow b(t - i0, \alpha(t + i0))$ are Hölder continuous with index $\mu \in (0, \frac{1}{2})$. The relation (75) then implies

$$\text{Im } \beta(t + i0) = c_1(t - 4)^{\sigma - \text{Re } \alpha(t + i0)} \sin(\log(t - 4) \cdot \text{Im } \alpha) + \mathcal{O}\left((t - 4)^{\sigma + \frac{1}{2}}\right), \quad (79)$$

and (78) follows from the threshold behaviour of $\alpha(t)$. \square

Remark 6 *The threshold behaviour (78) can be derived from equation (12) of [BZ62]. The statement of Lemma 1 is therefore a general consequence of elastic unitarity.*

For $4 < t < 5$ the logarithm is negative and diverges to minus infinity if $t \rightarrow 4$. Therefore the first term dominates near $t = 4$. Since we assume that $\text{Im } \alpha(t + i0)$ is positive, there is an interval $(4, t_1)$, $t_1 > 4$, such that $\text{Im } \beta(t + i0)$ is strictly negative for $4 < t < t_1$. The inequality $\text{Im } \beta < 0$ near threshold implies that the partial wave coefficients of β are negative for large angular momenta. Hence the techniques of section 5.2 cannot be applied to prove the inequalities (14) for the fixed point solutions.

Despite of this negative statement there remains a chance to derive these inequalities. Since the amplitude (68) satisfies elastic unitarity, there exist counterterms which modify the amplitude near the t -channel threshold $t = 4$ in order to achieve the correct support of the double spectral function $\rho(s, t)$. Moreover the double spectral function has to be positive in the Mahoux-Martin domain. The problems with $\text{Abs}_s R(t, s) \simeq \beta(t)s^{\alpha(t)}$ arise exactly in a region, where such compensations take place. With some more efforts it should be possible to prove the existence of a solution of the fixed point equations of section 6.1, which satisfies positivity for all angular momenta and all energies. There is even a chance to find a Regge amplitude which satisfies all constraints of section 2 and which has an asymptotically constant total cross section. But to obtain amplitudes with increasing total cross sections one has to develop new techniques.

A Estimates of partial wave amplitudes

The estimates of this Appendix are based on calculations in [Kup71, Kup82].

A.1 General statements

The Legendre functions of second kind $Q_l(z)$, $l = 0, 1, 2, \dots$, satisfy the relations

$$\begin{aligned} 0 < Q_{l+1}(z) < Q_l(z) < z^{-l} Q_0(z) \leq Q_0(z) = \frac{1}{2} \log \frac{z+1}{z-1}, \quad z > 1, \\ Q_l(z+x) < \frac{z}{z+x} Q_l(z) < Q_l(z), \quad z > 1, \quad x > 0, \\ (Q_l(z))^2 \leq 3z Q_0(z) Q_l(2z^2 - 1), \quad z > 1, \\ \lim_{z \rightarrow 1+0} Q_l(z)/Q_0(z) = 2^{-1}. \end{aligned} \tag{80}$$

If $z = 1 + 2(s-4)^{-1}(t+v)$, $s > 4$, $t > 0$, $v \geq 0$, then $2z^2 - 1 > 1 + 2(s-4)^{-1}(t + \tau_v(s)) + 16v(s-4)^{-2}t$ follows with

$$\tau_v(s) := 4v + 4 \frac{v^2}{s-4}. \tag{81}$$

The function $t = \tau_4(s)$, $s > 4$, is the boundary of the double spectral domain as determined by elastic unitarity. To evaluate the partial wave amplitudes it is convenient to introduce the following functions $\Phi_l^\mu(v; s)$, $l = 0, 1, 2, \dots$, $s \geq 4$, depending on the parameters $v \in [4, \infty)$ and $\mu > 0$

$$\begin{aligned} \Phi_l^\mu(v; s) &= 0 \quad \text{if } s = 4, \\ \Phi_l^\mu(v; s) &= \int_v^{v+1} (t-v)^\mu Q_l \left(1 + \frac{2t}{s-4} \right) dt = \int_0^1 t^\mu Q_l \left(1 + 2 \frac{t+v}{s-4} \right) dt \quad \text{if } s > 4. \end{aligned} \tag{82}$$

As a consequence of (80) these functions satisfy the relations

$$0 \leq \Phi_{l+1}^\mu(w; s) < \Phi_l^\mu(w; s) < \Phi_l^\mu(v; s) \quad \text{if } w > v \geq 4 \text{ and } s > 4. \tag{83}$$

The last of these inequalities has the following generalization: If $w > v \geq 4$, then for any pair of parameters $\mu > 0$, $\sigma > 0$ there is a constant $c_{\mu\sigma} < \infty$ such that

$$\Phi_l^\mu(w; s) < c_{\mu\sigma} \Phi_l^\sigma(v; s). \quad (84)$$

For s near threshold we have

$$\Phi_l(v; s) \sim (s - 4)^{l+1} \quad \text{if } s \rightarrow 4; \quad (85)$$

and for large s the functions $\Phi_l(v_1; s)$ increase like $\log s$, more precisely

$$\lim_{s \rightarrow \infty} \frac{\Phi_l(v; s)}{\log s} = 2^{-1}(1 + \mu)^{-1}. \quad (86)$$

Using Schwarz' inequality we obtain $(\Phi_l(v; s))^2 \leq (1 + \mu)^{-1} \int_0^1 t^\mu Q_l^2 \left(1 + 2\frac{t+v}{s-4}\right) dt$. Then the inequalities (80) imply the following estimates for the functions Φ_l , uniformly in $l = 0, 1, 2, \dots$,

$$(\Phi_l(4; s))^2 \leq c \cdot \frac{s-4}{s} \cdot \Phi_l(20; s) \quad \text{if } 4 \leq s \leq 20, \quad (87)$$

or, more generally,

$$(\Phi_l(v; s))^2 \leq c \cdot \frac{s-4}{s} \log s \cdot \Phi_l(\tau_v(w); s) \quad \text{if } 4 \leq s \leq w < \infty, \quad (88)$$

and

$$(\Phi_l(v; s))^2 \leq c \cdot \frac{s-4}{s} \log s \cdot \Phi_l(4v; s) \quad \text{if } s \geq 4. \quad (89)$$

The Froissart-Gribov integral (12) can be evaluated with the help of the following Lemma.

Lemma 1 *Let $h(t)$ be an integrable complex function on the interval $[v, \infty)$ with the properties*

$$\begin{aligned} |h(t)| &\leq c_1 (t - v)^\mu \quad \text{for } t \in [v, v + 1] \quad \text{with } \mu > 0, \\ |h(t)| &\leq c_2 t^\alpha (\log t)^{-\delta} \quad \text{with } \alpha > -1 \text{ and } \delta \geq 0, \end{aligned}$$

where $c_{1,2}$ are positive constants, then there exists a constant $c > 0$ such that

$$\left| \int_v^\infty h(t) Q_l \left(1 + \frac{2t}{s-4}\right) dt \right| \leq \begin{cases} c \cdot s^{\alpha+1} (\log s)^{-\delta} \Phi_l^\mu(v; s) & \text{if } \alpha > -1 \\ c \cdot \Phi_l^\mu(v; s) & \text{if } \alpha = -1 \text{ and } \delta > 1 \end{cases} \quad (90)$$

is valid for all $s > 4$ and $l \geq \max\{0, \alpha\}$.

The proof of this Lemma follows from Appendix B in [Kup71]. Two important estimates for the partial wave amplitudes are again formulated as Lemmata.

Lemma 2 *Let $F(s, t)$ be an amplitude which has the properties 1. – 3. and 6. of section 2. The cuts in t and u start at $t \geq t_1 \geq 4$ and $u \geq t_1$. Then the following statement is true: For any finite energy $s_1 \in [4, \infty)$ we can find constants $c_{1,2} \geq 0$ such that the partial wave amplitudes of F are bounded by*

$$|f_l(s)| \leq c_1 \cdot \left(\frac{s-4}{s}\right)^{-\frac{1}{2}} \Phi_l^\mu(t_1; s), \quad (91)$$

$$|\text{Im } f_l(s)| \leq c_2 \cdot \Phi_l^\mu(t_1; s) \quad (92)$$

and

$$\text{Im } f_l(s) - 2|f_l(s)|^2 \geq -c_3 \cdot s^\gamma \Phi_l^\mu(t_1; s) \quad (93)$$

for $l = 0, 1, 2, \dots$ and $4 \leq s \leq s_1$.

Proof. By assumption the amplitude F is polynomially bounded, and we can use the Froissart-Gribov integral for sufficiently large angular momenta, say $l > n$, for all energies $s \geq 4$. The partial wave amplitudes for $l \leq n$ are calculated with the integral (11). Using the Rodrigues' formula the integral (11) yields $|f_l(s)| \leq c(l) \cdot \left(\frac{s-4}{s}\right)^{l+\frac{1}{2}}$, $l = 0, 1, 2, \dots$, $s \geq 4$, with l -dependent constants $c(l)$. This result agrees with the threshold behaviour of (91), see (85). For the finite number of angular momenta $l = 0, 1, \dots, n$, and the finite energy range $4 \leq s \leq s_1$ the l -dependent constants $c(l)$ can be absorbed into (91). For $l > n$ and $4 \leq s \leq s_1$ the estimate (90) implies a bound $|f_l(s)| \leq c_1 \cdot \sqrt{\frac{s}{s-4}} \Phi_l^\mu(t_1; s)$ with μ being the Hölder index of the amplitude. Hence (91) is valid for $4 \leq s \leq s_1$. For the imaginary part (92) we get the additional threshold factor $\sqrt{\frac{s-4}{s}}$ of the absorptive part, see (15). The square $|f_l(s)|^2$ can be calculated with the help of (89) and (83). That leads to the lower bound (93). ■

Lemma 3 *Let $F(s, t)$ be an amplitude which has the properties 1. – 3. and 6. of section 2.. The cuts in t and u start at $t \geq t_1 \geq 4$ and $u \geq t_1$. If F is bounded by*

$$|F(s + i0, t)| \leq \text{const} \cdot \sum_{j=1,2} s^{\beta_j} (1 + |t|)^{\alpha_j} \quad (94)$$

for $s \geq s_1 \geq 4$ with $\alpha_j > -1$ and $\beta_j \in \mathbb{R}$, then there exists a constant $c_1 > 0$ such that the partial wave amplitudes f_l of F have the upper bound

$$|f_l(s)| \leq c_1 \cdot \sqrt{\frac{s}{s-4}} s^\gamma \Phi_l^\mu(t_1; s), \quad l = 0, 1, 2, \dots, \quad s \geq 4, \quad (95)$$

with $\gamma = \max_j \{\alpha_j + \beta_j\}$. If $\gamma < 0$ and $0 < \mu \leq \frac{1}{2}$ then there exists a constant $c_2 > 0$ such that the inequalities

$$\text{Im } f_l(s) - 2|f_l(s)|^2 \geq -c_2 \cdot s^\gamma \Phi_l^\mu(t_1; s), \quad l = 0, 1, 2, \dots, \quad (96)$$

are true for $s \geq 4$.

Proof. By assumption the amplitude F is polynomially bounded, and we can use the Froissart-Gribov integral for sufficiently large angular momenta $l > n$ for all energies. The partial wave amplitudes for $l \leq n$ are calculated with the integral (11).

For the finite energy range $4 \leq s \leq \max\{5, s_1\}$ the estimate (95) follows from Lemma 2. If $s \geq \max\{5, s_1\}$ the integral (11) yields the uniform bound $|f_l(s)| \leq c_1 \cdot s^\gamma$ for all $l = 0, 1, 2, \dots$. This bound together with the estimate derived for the partial wave amplitudes $f_l(s)$, $l > n$, with Lemma 1 imply the upper bound (95) for $s \geq \max\{5, s_1\}$.

The imaginary parts of f_l has the additional threshold factor $\sqrt{\frac{s-4}{s}}$ of the estimate (92). The square $|f_l(s)|^2$ can be calculated with the help of (89) and (83). That leads to the lower bound (96). ■

A.2 Amplitudes with positive spectral functions

Estimates of the partial wave amplitudes of a Mandelstam representation with positive spectral functions are needed for the construction of amplitudes, which satisfy the inelastic unitarity inequalities (14) for all energies.

It is straightforward to derive a precise estimate of the partial wave amplitudes of $A(s, t) = \text{Sym } F(s, t)$, where F satisfies the Mandelstam representations (20) or (21), if the spectral functions have the following properties:

- The double spectral function $\rho(x, y)$ is Hölder continuous with index $\mu \in (0, \frac{1}{2}]$. It has the structure

$$\begin{aligned} \rho(s, t) &= \psi_1(s, t) + \psi_2(t, s) \quad \text{with} \\ 0 \leq \psi_{1,2}(s, t) &\leq c \cdot t^\alpha s^{-1} (\log t)^{-\delta} (\log s)^{-\delta} \end{aligned} \quad (97)$$

with $-1 < \alpha \leq 1$, $\delta > 1$. The support of $\psi_{1,2}(s, t)$ lies within $[v, \infty) \times [w, \infty)$ with $4 \leq v \leq w \leq 20$.

In the sequel we simply write ψ instead of ψ_1 and ψ_2 . In the case of the crossing symmetry (6) of neutral pions we anyhow have $\psi_1(s, t) = \psi_2(s, t)$. But the following estimates are also valid for isospin-1 pions. If $-1 < \alpha \leq 0$ we can use the unsubtracted Mandelstam representation (20). If $0 < \alpha \leq 1$ we have to take the subtracted Mandelstam representation (21) which has an additional single spectral function with the properties:

- The single spectral function $\varphi(s)$ is Hölder continuous with index $\mu \in (0, \frac{1}{2}]$, positive and bounded by $s^{\alpha-1} (\log s)^{-\delta}$, $0 < \alpha \leq 1$, $\delta > 1$, for large s . It has a threshold behaviour $\varphi(s) \geq c(s-v)^\sigma$ with $c > 0$ for $v \leq s \leq v+1$, $v \geq 4$. The exponent is $\sigma = \frac{1}{2}$ if $v = 4$, and $\sigma = \mu \leq \frac{1}{2}$ if $v > 4$.

The Hilbert transform in (20) or (21) introduce additional $\log s$ and $\log t$ factors. The partial wave (11) $a_0(s)$ has the bound $|a_0(s)| \leq c \sqrt{\frac{s-4}{s}} s^{\alpha-1} (\log s)^{-\delta+2}$. All higher partial wave amplitudes can be estimated with the Froissart-Gribov integral (12) using Lemma 1. The final result – including all crossed terms – is

$$|a_l(s)| \leq c \sqrt{\frac{s}{s-4}} s^{\alpha-1} (\log s)^{-\delta+1} \Phi_l^\mu(v; s), \quad l = 0, 1, 2, \dots, s > 4. \quad (98)$$

In the case of one subtraction we have $0 < \alpha \leq 1$, in the case without subtraction the value of α is $-1 < \alpha \leq 0$. The relations (88) and (89) imply the upper bounds

$$|a_l(s)|^2 \leq \begin{cases} c \cdot \Phi_l^\mu(20; s) & \text{if } 4 \leq s \leq 20, \\ c \cdot s^{2\alpha-2} (\log s)^{-2\delta+2} \Phi_l^\mu(4v; s) & \text{if } s \geq 4. \end{cases} \quad (99)$$

For a lower bound on the imaginary part we need a more detailed knowledge about the behaviour of the double spectral function near the boundary of its support. If the support of $\psi(s, t)$ starts at $s = v$ and $t = w$, and ψ is bounded from below by

$$\psi(s, t) \geq c \left(\frac{s-v}{s} \right)_+^\sigma \left(\frac{t-w}{t} \right)_+^\mu t^\alpha s^{-1} (\log t)^{-\delta} (\log s)^{-\delta} \quad \text{if either } v \leq s \leq v+1 \text{ or } w \leq t \leq w+1, \quad (100)$$

then the Froissart-Gribov integral implies that the imaginary parts of the partial wave amplitudes with $l > 0$ have the lower bound

$$\left(\frac{s-4}{s} \right)_+^{\frac{1}{2}} \text{Im } a_l(s) \geq c_1 \left(\frac{s-v}{s} \right)_+^\sigma s^{-2} (\log s)^{-\delta} \Phi_l^\mu(w; s) + c_2 \left(\frac{s-w}{s} \right)_+^\mu s^{\alpha-1} (\log s)^{-\delta} \Phi_l^\sigma(v; s) \quad (101)$$

for $s \geq 4$. Here c and $c_{1,2}$ are positive constants. This lower bound is also correct for $l = 0$.

Since $\Phi_l^\mu(v; s) \sim \log s$ for large s , the partial wave amplitudes have the asymptotic behaviour $|\text{Re } a_l(s)| \lesssim s^{\alpha-1} (\log s)^{-\delta+2}$ and $\text{Im } a_l(s) \sim s^{\alpha-1} (\log s)^{-\delta+1}$. This statement implies

Corollary 1 *For increasing amplitudes with $\alpha = 1$ the inequalities $|a_l(s)|^2 \leq \text{Im } a_l(s)$ can be derived only if $\delta \leq 3$. To obtain values $\delta < 3$ one needs an improved estimate for $\text{Re } a_l(s)$.*

Now we take thresholds $v = 4$ and $4 \leq w \leq 20$ and choose parameters $\alpha < 1$ and $\delta > 1$ or $\alpha = 1$ and $\delta \geq 3$. The estimates (99) and (101) imply that the partial wave amplitudes satisfy the inequalities

$$\lambda \cdot \text{Im } a_l(s) - |a_l(s)|^2 \geq 0, \quad s \geq 4, \quad l = 0, 1, 2, \dots \quad (102)$$

if the multiplier $\lambda > 0$ is large enough. Then the partial wave amplitudes of $\lambda^{-1}A(s, t)$ fulfil the unitarity relations (14) for $s \geq 4$. Choosing a larger value of λ we get

$$\lambda \cdot \text{Im } a_l(s) - |a_l(s)|^2 \geq \begin{cases} c_1 \cdot \Phi_l^\mu(w; s) & \text{if } 4 \leq s \leq w + 1, \\ c_2 \cdot s^{\alpha-1}(\log s)^{-\delta} \Phi_l^\mu(v; s) & \text{if } s \geq v + 1 \end{cases} \quad (103)$$

$$\geq c \cdot s^{\alpha-1}(\log s)^{-\delta} \Phi_l^\mu(w; s) \quad \text{if } s \geq 4. \quad (104)$$

for $l = 0, 1, 2, \dots$ with constants $c_{1,2} > 0$ and $c > 0$ such that the partial wave amplitudes of $(2\lambda)^{-1}A(s, t)$ satisfy the relations

$$\text{Im } a_l(s) - 2|a_l(s)|^2 \geq c_0 \cdot \left(\frac{s-4}{s} \right)^{\mu-\frac{1}{2}} s^{\alpha-1}(\log s)^{-\delta} \Phi_l^\mu(w; s) \quad (105)$$

for $s \geq 4$ and $l = 0, 1, 2, \dots$, where $c_0 > 0$ is a strictly positive constant.

Remark 7 *In the case of unsubtracted amplitudes we can write down amplitudes with partial wave amplitudes, which behave like (98) and (101) for all $l = 0, 1, 2, \dots$, where the parameter α lies in the interval $-2 < \alpha < 0$. See the end of section 2.2 of [Kup71].*

The double spectral function $\omega(s, t)$ of the inhomogeneous part $B(s, t)$ in section 4.2 has thresholds at $v = w = 16$. If $\omega(s, t)$ has a lower bound (100) (with $\alpha \leq 0$ in the case without subtraction), the imaginary parts of the partial wave amplitudes $b_l(s)$ of B have the lower bounds

$$\text{Im } b_l(s) \geq c_1 \left(\frac{s-16}{s} \right)_+^\mu s^{\alpha-1}(\log s)^{-\delta} \Phi_l^\mu(16; s) \quad \text{if } s \geq 4, \quad (106)$$

and they satisfy the inequalities

$$\text{Im } b_l(s) \geq c_2 |b_l(s)|^2 \quad \text{if } s \geq 17. \quad (107)$$

These inequalities are sufficient to prove that – after an appropriate scaling of the inhomogeneous part ω – the fixed point problem of section 4.2 yields amplitudes, which also satisfy the unitarity inequalities (14) for all energies $s \geq 16$.

B Calculations for the Regge pole amplitudes

B.1 Functions with positive Legendre/Taylor coefficients

In section 5.1 the set $\mathcal{A}(s_1)$, $s_1 \geq 4$, has been defined as the set of functions $\phi(s, t)$, which have positive coefficients in their Legendre (partial wave) expansion for all energies $s \geq s_1$. In this Appendix this set will be called $\mathcal{A}_1(s_1)$. For further calculations it is useful to define a subclass $\mathcal{A}_2(s_1) \subset \mathcal{A}_1(s_1)$, which is characterized by positive coefficients in the Taylor expansion with respect to the variable $z = 1 + 2(s-4)^{-1}t$.

A function $\phi(s, t)$ is an element of the set $\mathcal{A}_2(s_1)$, $s_1 \geq 4$, if it has the following properties:

1. The function $\phi(s, t)$ is Hölder continuous in $s \geq s_1 \geq 4$ and holomorphic in the variable t in the cut plane \mathbb{C}_{cut} .

2. The function $\phi(s, t)$ is real if $s \geq s_1$ and $-s < t < 4$.
3. The power series expansion $\phi(s, t) = \sum c_n(s)x^n$ with the variable $x = \frac{s-4}{2}z = t + \frac{s-4}{2}$ has positive coefficients $c_n(s) \geq 0$ for $n = 0, 1, 2, \dots$ and $s \geq s_1$.

In [Kup71] the class \mathcal{A}_2 has been called \mathcal{A}' . Since $P_n(z) = \sum_{k=0}^n a_k z^k$ with positive coefficients a_k we have $\mathcal{A}_2(s_1) \subset \mathcal{A}_1(s_1)$; and the inclusions $\mathcal{A}_{1,2}(s_2) \subset \mathcal{A}_{1,2}(s_1)$ if $s_1 \leq s_2$ are obvious. The algebraic structures of the spaces $\mathcal{A}_1(s_1)$ and $\mathcal{A}_2(s_1)$ are similar:

From $\phi_{1,2}(s, t) \in \mathcal{A}_k$, $k = 1, 2$, we get

$$\alpha\phi_1(s, t) + \beta\phi_2(s, t) \in \mathcal{A}_k \quad \text{if } \alpha, \beta \geq 0 \quad (108)$$

$$\phi_1(s, t) \cdot \phi_2(s, t) \in \mathcal{A}_k, \quad (109)$$

and $\phi(s, t) \in \mathcal{A}_k$ implies $\exp(\lambda\phi(s, t)) \in \mathcal{A}_k$ for all parameters $\lambda \geq 0$.

Let $f(t)$ be a real function, which is \mathcal{L}^p -integrable over the interval $4 \leq t < \infty$ for some $p \in (1, \infty)$. Then the analytic function $F(t) = \int_4^\infty f(t') (t' - t)^{-1} dt'$ has a well defined power series expansions $F(t) = \sum_{n=0}^\infty c_n^F(s)x^n$ with coefficients

$$c_n^F(s) = \frac{1}{\pi} \int_4^\infty f(t) \left(\frac{s-4}{2} + t \right)^{-n-1} dt \quad (110)$$

If $f(t) \geq 0$ then these coefficients are positive, and the function $F(t)$ is an element of $\mathcal{A}_2(4)$. As a consequence the trajectory function (44) $\alpha(t)$ is an element of $\mathcal{A}_2(4)$ if $\alpha(\infty) \geq 0$. In the general case we have $\alpha(t) = \alpha_0 + \alpha_1(t)$ with $\alpha_0 \leq 0$ and $\alpha_1(t) \in \mathcal{A}_2(4)$.

By power series expansion we obtain

$$-\log(t_1 - t) + c \log(u_1 - u) \in \mathcal{A}_2(4) \quad (111)$$

where $u = 4 - s - t$, and the parameters are restricted to $4 \leq t_1 \leq u_1$ and $-1 \leq c \leq 1$. As a consequence of (111) and the rules (108) and (109) we obtain the following examples, which are needed for section 5.2.1. Thereby the function $\gamma_1(t)$ is an element of $\mathcal{A}_2(4)$, and $\gamma(t)$ is the sum $\gamma(t) = -\delta + \gamma_1(t)$ with a constant $\delta \geq 0$.

$$\begin{aligned} (s - s_1)^{\gamma(t)} &= (s - s_1)^{-\delta} \exp(\gamma_1(t) \log(s - s_1)) \in \mathcal{A}_2(s_1 + 2), \quad s_1 \geq 4, \\ (t_1 - t)^{-2\delta - \gamma(t)} &= \exp(-(\delta + \gamma_1(t))(\log(t_1 - t))) \in \mathcal{A}_2(4), \\ (t_1 - t)^{-\delta} (u_1 - u)^{-\delta} &\in \mathcal{A}_2(4), \\ (t_1 - t)^{-\gamma_1(t)} (u_1 - u)^{\gamma_1(t)} &\in \mathcal{A}_2(4), \\ (t_1 - t)^{-\gamma(t) - 2\delta} (u_1 - u)^{\gamma(t)} &\in \mathcal{A}_2(4). \end{aligned}$$

These results can be extended to integrals with positive weight functions. Let $\sigma(t) \geq 0$ be a positive integrable function with support inside the interval $17 \leq t \leq 18$. Then we have $\int \sigma(s')(s - s')^{\gamma(t)} ds' \in \mathcal{A}_2(20)$ and

$$N_1(s, t) := \int \int dt' du' \sigma(t') \sigma(u') (t' - t - 1)^{-\gamma(t) - 2\delta} (u' - 4 + s + t)^{\gamma(t)} \in \mathcal{A}_2(4). \quad (112)$$

This result has an important consequence for section 5.2.1. If we factorize the residue function into $\beta(t) = \beta_1(t) \cdot \beta_0(t)$ with $\beta_0(t) = \int dt' \sigma(t') (t' - t - 1)^{-\gamma(t) - 2\delta} \in \mathcal{A}_2(4)$, then the imaginary part (49) of the Regge ansatz is the product

$$\text{Im } \hat{R}(s + i0, u) = \beta_1(t) \cdot \int ds' \sigma(s') (s - s')_+^{\gamma(t)} \cdot N_1(s, t). \quad (113)$$

Hence choosing $\beta_1(t)$ as element of $\mathcal{A}_2(4)$ or of the larger class $\mathcal{A}_1(4)$ the product (113) is an element of $\mathcal{A}_1(20)$ as stated in section 5.2.1.

Remark 8 If the trajectory $\alpha(t)$ enters the half plane $\{l \mid \text{Re } l < 0\}$ for $t < 0$, the factor $(\sin \pi \gamma(t))^{-1}$ in (46) has poles in the physical region, and one needs zeros of $\beta_1(t)$ to compensate these poles. Such functions $\beta_1(t)$ do not exist within $\mathcal{A}_2(4)$ but in the larger class $\mathcal{A}_1(4)$, see Appendix D of [Kup71]. For the proof of the unitarity inequalities for the amplitudes, which saturate the Froissart bound, one has also to work with functions of the class \mathcal{A}_1 , see [KP79, Kup82].

For the comparison of real analytic functions, which are defined by dispersion integrals, the following Lemma is useful.

Lemma 4 Let $f(t)$ and $g(t)$ be two real functions on the interval $4 \leq t < \infty$, which are \mathcal{L}^p -integrable with $1 < p < \infty$. Assume that these functions satisfy the following restrictions:

- a) The function $f(t)$ is positive, $f(t) \geq 0$, and the threshold $t = 4$ belongs to the support of $f(t)$.
- b) There exists a constant $c_1 > 0$ such that $|g(t)| \leq c_1 f(t)$ in an interval $4 \leq t \leq t_1$ and for large t , $t > t_2 \geq t_1$.

Then there exists a constant $c_2 \geq 0$ such that the power series coefficients (110) of $F(t) = \int_4^\infty f(t') (t' - t)^{-1} dt'$ and $G(t) = \int_4^\infty g(t') (t' - t)^{-1} dt'$ satisfy $|c_n^G(s)| \leq c_2 \cdot c_n^F(s)$ for $n = 0, 1, 2, \dots$ and $s > 4$.

The proof follows from the representation (110); see Appendix D of [Kup71].

B.2 Unitarity of the Regge ansatz

For the constructions presented here the trajectories $\alpha(t)$ and the residue functions $\beta(t)$ have the following properties, which are more restrictive than those given in section 5.2; for the general case see [Kup71].

- a) The trajectory $\alpha(t)$ is real analytic and satisfies the dispersion relation (44). The imaginary part is Hölder continuous and strictly positive $\text{Im } \alpha(x + i0) > 0$ for $x > 4$. At threshold the imaginary part is bounded by $\text{Im } \alpha(t + i0) \leq c\sqrt{t-4}$ if $4 \leq t \leq 5$ with $c > 0$.
- b) The values of $\alpha(t)$ are restricted by $\alpha(0) \leq 1$ and $0 < \alpha(\infty) < \alpha(t) < 2$ if $t \leq 4$.
- c) The residue function $\beta(t)$ factorizes into

$$\beta(t) = \beta_1(t) \cdot \beta_0(t) \text{ with } \beta_0(t) = \int dt' \sigma(t') (t' - t - 1)^{-\gamma(t)}. \quad (114)$$

The convolution is performed with the same function $\sigma(t)$ as used in (46). The function $\beta_1(t)$ is real analytic with a cut at $t \geq 4$ and a Hölder continuous absorptive part. It has positive partial wave amplitudes and $\beta_1(t)$ is bounded by $|\beta_1(t)| \leq \text{const } (1 + |t|)^{-\delta}$, $t \in \mathbb{C}_{\text{cut}}$, with $\delta > \frac{1}{2}$.

The restriction $\alpha(\infty) > 0$ in b) allows only trajectories which stay in the right half plane $\text{Re } l > 0$ below threshold $t = 4$. This assumption simplifies the subsequent arguments. The factor $\beta_0(t)$ in (114) is an element of the class $\mathcal{A}_2(4)$, see Appendix B.1. With $\beta_1(t) \in \mathcal{A}_1(4)$ the residue function (114) has positive partial wave coefficients, as assumed in section 5.2.

As a consequence of these assumptions the Regge ansatz (46) $\hat{R}(s, u)$ has the upper bound

$$|\hat{R}(s, u)| \leq c \cdot (1 + |t|)^{-\theta} (1 + |s|)^{\varpi(t)} (1 + |s - t|)^{\varpi(t)}, \quad (s, t) \in \mathbb{C}_{\text{cut}}^2, \quad (115)$$

with $\varpi(t) = \text{Re } \gamma(t + i0)$ and an exponent $\theta > 2^{-1} (1 + \alpha(\infty))$. For fixed t the crossed contributions $\hat{R}(s, t) + \hat{R}(t, u)$ decrease stronger than $s^{-\frac{1}{2}}$ if $s \rightarrow \infty$. The large s asymptotics is therefore

dominated by $\hat{R}(s, u)$. Since $\alpha(\infty) > 0$ the background contribution $G(s, t)$ is chosen to satisfy a once subtracted Mandelstam representation, see section B.3.

The estimate (115) implies the uniform bound (48) for the partial wave amplitudes. Since we are interested in trajectories with $\max_{t \geq 4} \operatorname{Re} \alpha(t + i0) > 1$ the Froissart-Gribov integral does not give good estimates for the l -dependence of the partial wave amplitudes at high energies. But within a finite energy range, say $4 \leq s \leq 20$, we obtain from Lemma 1 of Appendix A.1

$$|f_l(s)| \leq c_2 \sqrt{\frac{s}{s-4}} \Phi_l(4; s), \quad 4 \leq s \leq 20. \quad (116)$$

In the next step the linear unitarity relations (43) of section 5.1 are derived for the Regge ansatz (46). If $s \geq 18$ (and $4 - s \leq t \leq 0$) the imaginary part $N(s, t) := \operatorname{Im} R(s + i0, u)$ and the real part $M(s, t) := \operatorname{Re} R(s + i0, u)$ are related by, see (49) and (50),

$$M(s, t) = -\cot \pi \gamma(t) \cdot N(s, t), \quad (117)$$

and we have

$$N(s, t) = 0 \quad \text{if} \quad 4 \leq s \leq 17. \quad (118)$$

As a consequence of property b) of the trajectory the function $\cot \pi \gamma(t)$ is holomorphic for $t \in \mathbb{C} \setminus [4, \infty)$, and the imaginary part has the upper bound

$$|\operatorname{Im} \cot \pi \gamma(t + i0)| \leq c \cdot \operatorname{Im} \gamma(t + i0) \quad \text{if} \quad t > 4 \quad (119)$$

with some constant $c \geq 0$. The function $\beta(t)$ is now factorized into (114). Then (49) and (50) can be written as

$$\begin{aligned} N(s, t) &= \beta_1(t) \cdot \int ds' \sigma(s') (s - s')_+^{\gamma(t)} \cdot N_1(s, t) \quad \text{and} \\ M(s, t) &= -\beta_1(t) \cdot \cot \pi \gamma(t) \cdot \int ds' \sigma(s') (s - s')_+^{\gamma(t)} \cdot N_1(s, t) \end{aligned} \quad (120)$$

where $N_1(s, t) \in \mathcal{A}_2(4)$ is given by (112) with $\delta = 0$. The imaginary part $N(s, t)$ is an element of $\mathcal{A}_2(20)$. Moreover, with the help of Lemma 4 we can find a function $\beta_1(t) \in \mathcal{A}_2(4)$ with positive imaginary part such that

$$-c \cdot \beta_1(t) \prec -\beta_1(t) \cot \pi \gamma(t) \prec c \cdot \beta_1(t) \quad (121)$$

holds with some constant $c > 0$. Using (42) these relations imply that the partial wave amplitudes of $M(s, t)$ can be estimated by those of $N(s, t) \in \mathcal{A}(20)$

$$-c \cdot N(s, t) \prec M(s, t) \prec c \cdot N(s, t) \quad \text{if} \quad s \geq 20. \quad (122)$$

Following the arguments of section 5.1 the partial wave amplitudes of the Regge ansatz $R(s, u)$ satisfy the quadratic unitarity inequalities (40) for $s \geq 20$. For energies $4 \leq s \leq 17$ we have $\operatorname{Im} f_l(s) = 0$ and – using (116) and (87) – we obtain the upper bound $|f_l(s)|^2 \leq c \sqrt{\frac{s-4}{s}} \Phi_l(20; s)$. These results imply the lower bounds

$$\operatorname{Im} f_l(s) - c |f_l(s)|^2 \geq \begin{cases} -c_1 \cdot \Phi_l(20; s) & \text{if } 4 \leq s \leq 17, \\ -c_2 \cdot \Phi_l(4; s) & \text{if } 17 \leq s \leq 20, \\ 0 & \text{if } s \geq 20, \end{cases} \quad (123)$$

with some constants $c_{1,2} > 0$. The partial wave amplitudes of the crossed term $R(s, t)$ are $(-1)^l f_l(s)$, $l = 0, 1, 2, \dots$. Hence the partial wave amplitudes of the sum $R(s, u) + R(s, t)$ are $2f_l(s)$ if l is even, and 0 if l is odd. These partial wave amplitudes satisfy again an estimate of the type (123).

B.3 Crossing symmetry

Crossing symmetry, correct threshold behaviour and the inelastic unitarity inequalities (14) for all energies $s \geq 4$ can be incorporated with a method which has been developed in [Kup71, Kup82]. The main results can be summarized in the following Propositions.

Proposition 1 *Let $F(s, t) = F(s, u)$ be an amplitude, which is symmetric in t and u and which has the properties 1. – 3. and 6. of section 2. Assume there exists an energy $s_1 \geq 4$ such that $F(s, t)$ has the upper bound*

$$|F(s + i0, t)| \leq \text{const} \cdot \sum_{j=1,2} s^{\beta_j} (1 + |t|)^{\alpha_j}$$

for $s \geq s_1 \geq 4$ with $\alpha_j > -1$ and $\alpha_j + \beta_j < 0$ (or $\alpha_j + \beta_j < -1$), $j = 1, 2$. Then one can find a constant $\lambda > 0$ and a crossing symmetric amplitude $G(s, t)$, which satisfies a Mandelstam representation with at most one (without) subtraction and with positive spectral functions, such that the following statement is true:

The sum $A(s, t) = \lambda F(s, t) + G(s, t)$ fulfils the unitarity inequalities $\text{Im } a_l(s) \geq |a_l(s)|^2$, $l = 0, 1, 2, \dots$ for all energies $s \geq 4$.

Proof. Following Lemma 3 in Appendix A.1 the partial wave amplitudes of $F(s, t)$ can be estimated by (96) with $\gamma = \max_j \{\alpha_j + \beta_j\} < 0$ and a constant $c_2 > 0$. Let $G(s, t)$ be a crossing symmetric amplitude, which has a Mandelstam representation with at most one subtraction and with positive spectral functions (as considered in Appendix A.2). We assume that the inequalities (105) for the partial wave amplitudes $\text{Im } g_l(s) - 2|g_l(s)|^2 \geq c_0 \cdot \left(\frac{s-4}{s}\right)^{\mu-\frac{1}{2}} s^{\alpha-1} (\log s)^{-\delta} \Phi_l^\mu(t_1; s)$ with α in the interval $\gamma + 1 < \alpha < 1$ are valid. Then the amplitude $A = \lambda F + G$ with $0 < \lambda \leq \min \{1, c_2^{-1} c_0\}$ has partial wave amplitudes which fulfil the constraints

$$\begin{aligned} \text{Im}(\lambda f_l + g_l) - |\lambda f_l + g_l|^2 &\geq \lambda \text{Im } f_l - 2\lambda^2 |f_l|^2 + \text{Im } g_l - 2|g_l|^2 \\ &\geq \lambda \text{Im } f_l - 2\lambda |f_l|^2 + \text{Im } g_l - 2|g_l|^2 \geq 0, \quad l = 0, 2, 4, \dots \end{aligned}$$

for $s \geq 4$. If $\gamma < -1$ then $\gamma + 1 < \alpha < 0$ is possible, and we can choose an amplitude G which satisfies an unsubtracted Mandelstam representation. ■

Proposition 2 *Let $F(s, t) = F(s, u)$ be an amplitude, which is symmetric in t and u and which has the properties 1. – 3. and 6. of section 2. Assume there exists an energy $s_1 \geq 4$ and a constant $\lambda_0 > 0$ such that the partial wave amplitudes of F satisfy the inelastic unitarity constraints $\text{Im } f_l(s) \geq \lambda_0 |f_l(s)|^2$ for $s \geq s_1$. Then we can find a constant $\lambda > 0$ and a crossing symmetric amplitude $G(s, t)$, which satisfies an unsubtracted Mandelstam representation with positive double spectral function, such that the sum $A(s, t) = \lambda F(s, t) + G(s, t)$ fulfils the unitarity inequalities $\text{Im } a_l(s) \geq |a_l(s)|^2$, $l = 0, 1, 2, \dots$ for all energies $s \geq 4$.*

Proof. The amplitude $F(s, t)$ is polynomially bounded and has a threshold behaviour (15). The partial wave amplitudes satisfy the estimates of Lemma 2 for $s \leq s_1$, hence

$$\begin{aligned} \text{Im } f_l - \lambda_0 |f_l|^2 &\geq \begin{cases} -c \cdot \Phi_l^\mu(t_1; s) & \text{if } 4 \leq s \leq s_1 \\ 0 & \text{if } s \geq s_1 \end{cases} \\ &\geq -c_\alpha \cdot s^{\alpha-1} \Phi_l^\mu(t_1; s) \quad \text{if } s \geq 4, \end{aligned}$$

where any constant $\alpha < 0$ is possible. Using the arguments of the proof for Proposition 1 we can find a constant $\lambda > 0$ and an amplitude $G(s, t)$, which is given by an unsubtracted

Mandelstam representation, such that the partial wave amplitudes $a_l = \lambda f_l + g_l$ satisfy the unitarity inequalities for all $s \geq 4$. ■

We now apply these Propositions to the Regge amplitudes of section 5. The $t-u$ symmetrized Regge contribution $F(s, t) := \hat{R}(s, u) + \hat{R}(s, t)$ satisfies the assumptions of Proposition 2. Hence we can find a constant $\lambda_1 > 0$ and a crossing symmetric unsubtracted Mandelstam integral $G_2(s, t)$ with positive double spectral function such that

$$A_1(s, t) = \lambda_1 \left(\hat{R}(s, u) + \hat{R}(s, t) \right) + G_1(s, t) \quad (124)$$

has partial wave amplitudes which satisfy the unitarity inequalities (14) for $s \geq 4$.

The crossed Regge term $\hat{R}(t, u)$ satisfies the assumptions of Proposition 1, see the bound (115) for $\hat{R}(s, u)$. Hence we can find a constant $\lambda_2 > 0$ and a crossing symmetric amplitude $G_2(s, t)$ as indicated in Proposition 1 such that

$$A_2(s, t) = \lambda_2 \hat{R}(t, u) + G_2(s, t) \quad (125)$$

has partial wave amplitudes, which satisfy the unitarity inequalities (14) for $s \geq 4$.

To derive unitarity for the sum (45) the statement of Remark 3 about the inelastic inequalities is essential. Starting from (124) and (125) with constants $\alpha_{1,2} \geq 0$ such that $\alpha_1 + \alpha_2 \leq 1$ and $\alpha_1 \lambda_1 = \alpha_2 \lambda_2 = c > 0$ we obtain

$$A(s, t) = \alpha_1 A_1(s, t) + \alpha_2 A_2(s, t) = c \left(\hat{R}(s, u) + \hat{R}(s, t) + \hat{R}(t, u) \right) + G(s, t), \quad (126)$$

is an amplitude which satisfies the properties 1.-3. and 5. of section 2. Thereby $G(s, t) = \alpha_1 G_1(s, t) + \alpha_2 G_2(s, t)$ is given by a Mandelstam representation without (if $\alpha(\infty) < 0$) or with one subtraction (if $\alpha(\infty) > 0$ as assumed in Appendix B.2). The constant c can be absorbed into the residue function $\beta(t)$ and (126) yields the representation (45).

C Khuri poles

C.1 Mellin transformation

Let $f(t)$ be a complex function with support in \mathbb{R}_+ such that $\int_0^\infty |f(t)t^{-\gamma}|^2 t^{-1} dt < \infty$ exists for some $\gamma \in \mathbb{R}$. Then the Mellin transformation [Tit48]

$$a(\nu) = \frac{1}{\pi} \int_0^\infty f(t)t^{-\nu-1} dt = \mathcal{M}[f(t)](\nu) \quad (127)$$

is defined at least for $\text{Re} \nu = \gamma$ with $\int_{-\infty}^\infty |a(\gamma + ix)|^2 dx = \frac{2}{\pi} \int_0^\infty |f(t)t^{-\gamma}|^2 t^{-1} dt$. The inverse Mellin transformation is given by [Tit48]

$$f(t) = \mathcal{M}^{-1}[a(\nu)](t) := \frac{1}{2i} \int_\gamma a(\nu)t^\nu d\nu. \quad (128)$$

The symbol $\int_\gamma d\nu$ means integration along the line $\nu = \gamma + ix$, $-\infty < x < \infty$.

Let \mathcal{L}_γ , $\gamma \in \mathbb{R}$, be the Hilbert space of all functions $f(t)$ with a finite norm (27), then the Mellin transform maps this space isometrically onto the Sobolev space $\mathcal{S}(\gamma)$ of functions $a(\nu)$ with norm

$$|a(\nu)|_\gamma = \left[\frac{\pi}{2} \int_{\mathbb{R}} dx \left(|a(\gamma + ix)|^2 + \left| \frac{d}{dx} a(\gamma + ix) \right|^2 \right) \right]^{\frac{1}{2}}. \quad (129)$$

If the support of $f(t) \in \mathcal{L}_\gamma$ lies inside $[t_0, \infty)$ with $t_0 > 0$, then the integral (127) exists also for $\text{Re } \nu > \gamma$ and defines a holomorphic function in that region. Moreover we have $a(\nu) \in \mathcal{S}(\gamma')$ for all $\gamma' \geq \gamma$. If $f(t) \in \mathcal{L}_\gamma$ with $-1 < \gamma < 0$, then the dispersion integral $F(t) = \frac{1}{\pi} \int_0^\infty f(t')(t' - t)^{-1} dt'$ exists, and we can calculate the Mellin transformation of $F(-t)$

$$\varphi(\nu) = \mathcal{M}[F(-t)] = \frac{1}{\pi} \int_0^\infty F(-t) t^{-\nu-1} dt \quad (130)$$

with the result $\varphi(\nu) = -a(\nu) (\sin \pi \nu)^{-1}$. The inverse Mellin transform then yields the Khuri representation (69) of the function $F(t)$

$$F(t) = -\frac{1}{2i} \int_\gamma \frac{a(\nu)}{\sin \pi \nu} (-t)^\nu d\nu. \quad (131)$$

If $a(\nu) \in \mathcal{S}(\gamma)$, where γ is not an integer, then the functions $(\sin \pi \nu)^{-1} a(\nu)$ and $(\sin \pi \nu)^{-1} \exp(\pm i \pi \nu) a(\nu)$ are also elements of $\mathcal{S}(\gamma)$, and the mappings $a(\nu) \rightarrow (\sin \pi \nu)^{-1} a(\nu)$ and $a(\nu) \rightarrow (\sin \pi \nu)^{-1} \exp(\pm i \pi \nu) a(\nu)$ are continuous.

The following example is used in Appendix C.2. Let $t_1 > 0$ be a positive number and $\alpha \in \mathbb{C}$ with $\text{Re } \alpha > -1$. Then the function $\mathbb{R} \ni t \rightarrow f(t) = (t^\alpha - t_1^{\alpha+1} t^{-1}) \Theta(t - t_1)$ is Hölder continuous, and it is an element of \mathcal{L}_γ for all $\gamma > \text{Re } \alpha$. The Mellin transform $a(\nu) = \mathcal{M}[f(t)](\nu)$ is calculated

$$a(\nu) = \frac{1}{\pi} \int_{t_1}^\infty (t^\alpha - t_1^{\alpha+1} t^{-1}) t^{-\nu-1} dt = \frac{1}{\pi} t_1^{\alpha-\nu} \left(\frac{1}{\nu - \alpha} - \frac{1}{\nu + 1} \right) = \frac{1}{\pi} t_1^{\alpha-\nu} \frac{\alpha + 1}{(\nu - \alpha)(\nu + 1)}, \quad (132)$$

if $\text{Re } \nu > \text{Re } \alpha$.

C.2 The pole ansatz

A Regge pole at $\nu = \alpha(s)$ with residue $\beta(s)$ leads to a series of Khuri poles at positions $\nu = \alpha(s) - n$, $n = 0, 1, 2, \dots$ with residues

$$r_n(s) = \frac{(4-s)^n}{n!} \frac{(-\alpha)_n (-\alpha)_n}{(-2\alpha)_n} \beta(s). \quad (133)$$

A suitable Regge ansatz, which includes $N + 1$ Khuri poles, is, see (132),

$$\begin{aligned} a_R(s, \nu) &= a_R[\alpha, \beta](s, \nu) := \beta t_1^{\alpha(s)-\nu} \left(\frac{1}{\nu - \alpha(s)} - \frac{1}{\nu + 1} \right) \\ &+ \beta t_1^{\alpha(s)-\nu} \sum_{n=1}^N \frac{1}{n!} \left(\frac{4-s}{t_1} \right)^n \frac{(-\alpha)_n (1+\nu)_n}{(1+n+2\nu)_n} \left(\frac{1}{\nu - \alpha + n} - \frac{1}{\nu + 1} \right). \end{aligned} \quad (134)$$

Thereby $t_1 > 16$ is a parameter. The function $\nu \rightarrow a_R(s, \nu)$ is an element of the spaces $\mathcal{S}(\gamma)$ for all $\gamma > -1$ with $\gamma \notin \{\text{Re } \alpha(s) - n \mid n = 0, 1, \dots, N\}$. Moreover, its inverse Mellin transform is Hölder continuous in the variable t .

Under the assumptions a) - c) of section 6 about the trajectory $\alpha(s)$ it is sufficient to choose $N = 2$ in (134). For large energies, say $|s| > s_2$, we have $\text{Re } \alpha(s) < \gamma_0$, $-1 < \gamma_0 < 0$. Then the Regge amplitude is defined for these energies by the integral (131)

$$R(s, t) = -\frac{1}{2i} \int_{\gamma_0} \frac{a_R(s, \nu)}{\sin \pi \nu} (-t)^\nu d\nu \quad (135)$$

with the t -channel absorptive part $R_t(s, t) = \mathcal{M}_\gamma^{-1} [a_R(s, \nu)]$ if $t \geq 4$. We can shift the contour of integration to $\text{Re } \nu = \gamma_1$ and have to collect the residues at $\nu = 0, 1, 2$,

$$R(s, t) = -\frac{1}{2i} \int_{\gamma_1} \frac{a_R(s, \nu)}{\sin \pi \nu} (-t)^\nu d\nu + a_R(s, 0) + a_R(s, 1)t + a_R(s, 2)t^2. \quad (136)$$

This formula is still correct when the Regge trajectory enters the region $\gamma < \text{Re} \alpha(s + i0) < \frac{3}{2}$. Adding the crossed term $R(s, u)$ we obtain

$$\begin{aligned} R(s, t) + R(s, u) &= -\frac{1}{2i} \int_{\gamma_1} \frac{a_R(s, \nu)}{\sin \pi \nu} [(-t)^\nu + (-u)^\nu] d\nu \\ &\quad + 2a_R(s, 0) + (4-s)a_R(s, 1) + (t^2 + u^2)a_R(s, 2). \end{aligned} \quad (137)$$

The sum $2a_R(s, 0) + (4-s)a_R(s, 1) + (t^2 + u^2)a_R(s, 2)$ may lead to singularities if a Regge pole crosses the integer values $\alpha = 0, 1, 2$. If $\alpha = 0$ such a singularity appears in $a_R(s, 0)$ unless the residue vanishes. To compensate the pole at $\alpha = 0$ the residue function $\beta(s)$ has to include a ghost killing factor $\alpha(s)$ as done in [Kup77]. If $\alpha = 1$ the Khuri pole at $\nu = 1$ and its daughter pole at $\nu = 0$ compensate. This is a kinematic pole killing due to the projection onto even partial waves in (137). The case $\alpha = 2$ is excluded by the assumptions on the trajectory function.

Remark 9 *If a Regge pole enters the strip $\gamma_0 + 1 < \text{Re} \alpha(s + i0) < \gamma_0 + 2$ an alternative formula for the Regge amplitude (136) is*

$$R(s, t) = -\frac{1}{2i} \int_{\gamma_0} \frac{a_R(s, \nu)}{\sin \pi \nu} (-t)^\nu d\nu + \frac{\pi r_0(s)}{\sin \pi \alpha(s)} (-t)^{\alpha(s)} - \frac{\pi r_1(s)}{\sin \pi \alpha(s)} (-t)^{\alpha(s)-1}. \quad (138)$$

The Mellin transform of $\text{Abs}_t A(s, t)$ is then given by

$$a(s, \nu) = a_R[\alpha, \beta](s, \nu) + b(s, \nu) \quad (139)$$

where $b(s, \nu)$ is a holomorphic background. The crossed terms $R(t, s) + R(t, u)$ and $R(u, s) + R(u, t)$ do only contribute to the background $b(s, \nu)$ if the residue function $\beta(s)$ decreases at least like $|s|^{-N-1}$.

C.3 The unitarity integral

If $A_t(s, \cdot) \in \mathcal{L}_{\gamma_0}$ with $-\frac{1}{2} + \mu < \gamma_0 < 0$ the Mellin transform $a(s, \nu) = \mathcal{M}[A_t(s, t)]$ is holomorphic in ν for $\text{Re } \nu > \gamma_0$, and it is an element of $\cap_{\gamma \geq \gamma_0} \mathcal{S}(\gamma)$. For such amplitudes the unitarity integral (31) is transformed into [Kup77]

$$w(s, \nu) = \sqrt{\frac{s-4}{s}} (s-4)^\nu I(\gamma_0, \gamma_0; s, \nu), \quad 4 \leq s \leq 16. \quad (140)$$

Thereby $w(s, \nu) = \mathcal{M}[\psi(s, t)]$ is the Mellin transform of the double spectral function, and $I(\gamma, \gamma'; s, \nu)$ is the Mellin-Barnes type integral

$$I(\gamma, \gamma'; s, \nu) := -\frac{1}{4\pi^2} \int_\gamma d\xi \int_{\gamma'} d\eta (s-4)^{\xi+\eta-2\nu} M(s, \nu, \xi, \eta) a(s+i0, \xi) a(s-i0, \eta). \quad (141)$$

with the kernel

$$M(s, \nu, \xi, \eta) = B(1+\xi, \nu-\xi) B(1+\eta, \nu-\eta) B(1+\nu, 1-\nu+\xi+\eta) \quad (142)$$

The function $B(x, y) = \Gamma(x)\Gamma(y) (\Gamma(x+y))^{-1}$ is the Euler beta function. The integral (143) is defined with integration along the lines $\text{Re } \xi = \gamma$ and $\text{Re } \eta = \gamma'$ such that $\gamma, \gamma' < \text{Re } \nu < 1 + \gamma + \gamma'$. If $a(s, \nu)$ is holomorphic for $\text{Re } \nu > \gamma \geq \gamma_0$, the unitarity identity (140) has an analytic continuation to

$$w(s, \nu) = \sqrt{\frac{s-4}{s}} (s-4)^\nu I(\gamma, \gamma; s, \nu), \quad 4 \leq s \leq 16, \quad (143)$$

which is valid within the strip $\gamma < \text{Re } \nu < 1 + 2\gamma$. Take ν in the strip $\gamma < \text{Re } \nu < \min\{\gamma + 1, 1 + 2\gamma\}$. By a shift of the contour of integration to the right to $\gamma + 1$ (such that $\gamma < \text{Re } \nu < \gamma + 1$) we obtain

$$\begin{aligned} \sqrt{\frac{s}{s-4}} (s-4)^{-\nu} w(s, \nu) &= B(1+\nu, 1+\nu) a(s+i0, \nu) a(s-i0, \nu) \\ &\quad + a(s+i0, \nu) \varphi_1(s-i0, \nu) + \varphi_1(s+i0, \nu) a(s-i0, \nu) \\ &\quad + I(\gamma+1, \gamma+1; s, \nu) \end{aligned} \quad (144)$$

with

$$\varphi_1(s \pm i0, \nu) = \frac{1}{2\pi i} \int_{\gamma+1} d\xi (s-4)^{\xi-\nu} B(1+\xi, \nu-\xi) B(1+\nu, 1+\xi) a(s \pm i0, \xi).$$

For $4 \leq s \leq 16$ we have $w(s, \nu) = (2i)^{-1} (a(s+i0, \nu) - a(s-i0, \nu))$. Assume that a pole $\beta(s+i0) (v - \alpha(s+i0))^{-1}$ of $a(s+i0, \nu)$ enters the strip $\gamma < \text{Re } \nu < \gamma + 1$, then at the residue the following identity follows from (144)

$$2i \sqrt{\frac{s-4}{s}} (s-4)^{\alpha(s+i0)} \varphi(s-i0, \alpha(s+i0)) = 1, \quad (145)$$

where

$$\varphi(s \pm i0, \nu) = B(1+\nu, 1+\nu) a(s \pm i0, \nu) + \varphi_1(s \pm i0, \nu) \quad (146)$$

is (up to a factor 2) the reduced partial wave amplitude of $A(s \pm i0, t)$. If $|v - \alpha(s-i0)| < 1$ we have

$$\varphi(s-i0, \nu) = \beta(s-i0) B(1+\nu, 1+\nu) (v - \alpha(s-i0))^{-1} + \phi(s-i0, \nu), \quad (147)$$

with a holomorphic “background” $\phi(s-i0, \nu)$, which originates from the daughter pole contributions, the background of $a(s-i0, \nu)$ and from $\varphi_1(s-i0, \nu)$. Since $\text{Im } \alpha$ is small, the number $\alpha(s+i0)$ lies in the neighbourhood of $\alpha(s-i0)$, and we can insert (147) into (145). The resulting identity

$$\beta(s-i0) = (s-4)^{\sigma-\alpha(s+i0)} B^{-1} \chi(s) - 2i \sqrt{\frac{s-4}{s}} (s-4)^\sigma \chi(s) B^{-1} \phi(s-i0, \alpha) \quad (148)$$

with $B^{-1} = B^{-1}(1+\alpha, 1+\alpha)$ has exactly the form (75), only the interpretation of $b(s, \nu)$ has changed. The function $\mathbb{R} \ni s \rightarrow B^{-1}(1+\alpha(s+i0), 1+\alpha(s+i0)) \phi(s-i0, \alpha(s+i0))$ is Hölder continuous.

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